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BAIRE CATEGORY THEORY IN ANALYSIS

A THESIS

Presented to

The Faculty of the Graduate Division

by

David Lowell Lovelady

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

April, 1968

BAIRE CATEGORY THEORY IN ANALYSIS

Approved:

Chairman

Date approved by Chairman: 5-5-88

To my friend, with a little
help from whom I get by.

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to Dr. Eric R. Immel, my thesis adviser, for his patience and expert guidance in the preparation of this thesis. I also wish to thank the other two members of my committee, Dr. H. A. Gersch and Dr. R. H. Kasriel, for their careful reading of the manuscript. Finally, my sincere thanks and appreciation to Mrs. Betty Sims, whose expert typing made publication of this thesis possible.

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CHAPTER I

INTRODUCTION

To say that a set D in a topological space X is dense is to say that D is "well-distributed" throughout X, in the sense that the closure of D is X. It is quite possible, however, to have two such well-distributed sets which do not intersect, e.g., the rationals and the irrationals in the set of reals. If, in addition, we require that two sets be not only dense but also open, in the space concerned, then they have open dense intersection. This result immediately generalizes to any finite collection of open dense sets, but does not, in general, extend to infinite intersections of open dense sets.

That countable intersections of open dense sets are dense in complete metric spaces is the content of the Baire category theorem, the primary tool of investigation in this paper. If we agree to call a set nowhere dense when its closure contains no non-empty open set, first category when it can be written as a countable union of nowhere dense sets, and second category when it is not first category, then an equivalent formulation of the Baire theorem is that every complete metric space is of second category. The Baire category theorem was proved for sets in the Euclidean line by R. Baire [1], and independently by W. Osgood [8] about the same time. The generalization to metric spaces is due to F. Hausdorff [7].

In Chapter III we apply category arguments to some examples in classical analysis. The most striking result here is the classification of continuous functions with respect to differentiability properties. The collection of all continuous real-valued functions on [0,1], with the supremum metric, is a complete metric space, and therefore of second category. It turns out that the set of all continuous real-valued functions on [0,1] which have a finite right-hand derivative at even one point is of first category. Thus there exist continuous nowhere-differentiable functions.

The Baire theorem and related theorems are particularly powerful in functional analysis. In linear topological spaces which are also complete metric spaces, it frequently suffices to find a non-empty open set with a particular property. Then the linearity of the space allows us to extend the results to the whole space. Using essentially this technique, we prove the uniform boundedness theorem and the open mapping theorem. We then use the open mapping theorem to prove Banach's closed graph theorem.

In a measure space, we can make the underlying σ-algebra into a pseudo-metric space by letting the distance between two sets be the measure of their symmetric difference. If we take equivalence classes to form a metric space, we turn up with a complete metric space. We can then use the Baire theorem to deduce information about limits of sequences of complex measures on the original measure space. This technique gives us, for example, the Vitali-Hahn-Saks theorem.

CHAPTER II

GENERAL THEORY AND SOME TOPOLOGICAL RESULTS

Section 1: Definitions and Elementary Relationships

There are numerous ways in which a set can be considered "small," one of the most absolute and least interesting of which is for it to be empty. In this paper we shall consider a concept in topological spaces which, speaking loosely, measures the relative emptiness of a set. Let X be a topological space and let D be a subset of X. Then we say D is *dense in X* if and only if $\bar{D} = X$, where the bar denotes closure. Since X is the largest open subset of itself, the extreme opposite of D being dense would be for $\bar{D}$ to contain no non-empty open set in X. Accordingly, we make the following definition.

II.1 Definition: Let X be a topological space and let N be a subset of X. Then N is said to be *nowhere dense in X* if and only if $\bar{N}$ contains no non-empty open subset of X.

To verify that there do exist non-trivial nowhere dense sets, we prove the following minor result.

II.2 Lemma: Let E^n be n-dimensional Euclidean space. Let A be a subset of E^n such that $\inf \{|x-y| : x \in A,\ y \in A,\ x \neq y\} > 0$. Then A is nowhere dense in E^n.

Proof: There exists $\alpha > 0$ such that $|x-y| > \alpha$ whenever $x \in A$, $y \in A$, and $x \neq y$. Suppose $p \in E^n$, $p \notin A$. We claim there exists $\varepsilon > 0$ such that $N(p,\varepsilon) \cap A = \phi$, where $N(p,\varepsilon)$ is the ε-neighbourhood about p.

Choose ε_1, $0 < \varepsilon_1 < \frac{1}{2}\alpha$. If $N(p,\varepsilon_1) \cap A = \phi$, our claim is correct, so suppose there exists $x \in N(p,\varepsilon_1) \cap A$. Pick $y \in A$, $y \neq x$. If $y \in N(p,\varepsilon_1)$, $|x-y| \leq |x-p| + |p-y| < \alpha$ which is impossible since $x \in A$, $y \in A$, and $y \neq x$. Now let $\varepsilon = \frac{1}{2}|p-x|$. Note that $\varepsilon > 0$ since $p \neq x$. We now have $N(p,\varepsilon) \cap A = \phi$, so p is not a limit point of A. Thus A is closed since A^c contains no limit points of A. It remains to show that A contains no non-empty open set. Suppose $\phi \neq U \subset A$, and U is open. Choose $x \in U$. There exists ε, $0 < \varepsilon < \alpha$, such that $N(x,\varepsilon) \subset U \subset A$. But this is impossible since if $y \in A$, $y \neq x$, then $|x-y| > \alpha > \varepsilon$. Thus U is not open. This completes the proof.

The relationship between nowhere dense sets and dense sets is not quite as transparent as at first it might seem. For example, a set can be dense and have dense complement. In E^n the set of all points with rational coordinates is certainly dense, as is its complement. There do exist, however, some worthwhile relationships which we tabulate here for easy reference. First we introduce an alternative characterization of denseness.

II.3 Theorem: Let X be a topological space, and let D be a subset of X. Then D is dense in X if and only if D intersects every non-empty open set in X.

Proof: Suppose D is dense in X and there exists open $V \neq \phi$ such that $V \cap D = \phi$. Now V^c is closed and $D \subset V^c$. Thus $\bar{D} \subset V^c$. But since $V \neq \phi$, $V^c \neq X$, so $\bar{D} \neq X$, a contradiction. Consequently, if D is dense and $V \neq \phi$ is open, $V \cap D \neq \phi$. Now suppose D intersects every non-empty open set V. Pick $x \in X$. Since every non-empty open set intersects D, every neighbourhood of x intersects D, and consequently $x \in \bar{D}$. But $x \in X$ was arbitrary; thus $X = D$ and D is dense. This completes the proof.

This characterization of denseness will be useful in establishing the following results.

II.4 Theorem: Let X be a topological space. Then:

(i) A finite intersection of dense open sets in X is dense in X.

(ii) A finite union of nowhere dense sets in X is nowhere dense in X.

(iii) If E is nowhere dense in X, E^c is dense in X.

(iv) If E is closed and E^c is dense in X, then E is nowhere dense in X.

Proof: (i) Let $V_1, V_2, \cdots, V_n$ be dense open sets and let $V = \bigcap_{i=1}^{n} V_i$. It suffices to show V intersects every non-empty open set of X. Let U be non-empty and open. Since V_1 is dense, $V_1 \cap U \neq \phi$. Since V_1 is open, $V_1 \cap U$ is open. Thus, $V_1 \cap U$ is a non-empty open set. Therefore, $V_2 \cap (V_1 \cap U) = \left(\bigcap_{i=1}^{2} V_i\right) \cap U$ is open and non-empty, since V_2 is open and dense. Repeating this procedure it follows that $\left(\bigcap_{i=1}^{k} V_i\right)$ U is open and non-empty for each $k \leq n$. In particular, $V \cap U = \left(\bigcap_{i=1}^{n} V_i\right) \cap U \neq \phi$.

(ii) Let $N_1, N_2, \cdots, N_m$ be nowhere dense sets, and let $N = \bigcup_{i=1}^{m} N_i$. Now $\bar{N} = \bigcup_{i=1}^{m} \bar{N}_i$. Suppose U is a non-empty open set, $U \subset \bar{N}$. Note that $(\bar{N})^c \subset U^c$. But $\bar{N} = \bigcup_{i=1}^{m} \bar{N}_i$; thus $\bigcap_{i=1}^{m} (\bar{N}_i)^c \subset U^c$. For each i, since $\bar{N}_i$ contains no non-empty open set, $(\bar{N}_i)^c$ intersects every non-empty open set. Also $(\bar{N}_i)^c$ is open since $\bar{N}_i$ is closed. Thus, by (i), $\bigcap_{i=1}^{m} (\bar{N}_i)^c$ is dense. Consequently, $(\bar{N})^c \cap U = \bigcap_{i=1}^{m} (\bar{N}_i)^c \cap U \neq \phi$. But $(\bar{N})^c \subset U^c$, so $(\bar{N})^c \cap U = \phi$. This is clearly a contradiction, so we cannot have a non-empty open set contained in $\bar{N}$. Thus N is nowhere dense.

(iii) Since E is nowhere dense, E contains no non-empty open set. Hence, E^c intersects every non-empty open set and E^c is dense.

(iv) Since E is closed, $E = \bar{E}$. Since E^c is dense, every non-empty open set intersects E^c, and thus no non-empty open set is contained in $E(=\bar{E})$. Consequently, E is nowhere dense. This completes the proof of the theorem.

In (ii) of II.4 we showed that a finite union of nowhere dense sets is nowhere dense. It turns out that this result does not generalize to infinite unions. To see this, we take the space to be E^1 and assume the rational numbers have been enumerated by $\{r_1, r_2, \cdots\}$. For each $n \geq 1$, let A_n be the one-point set $\{r_n\}$. Each A_n is nowhere dense, but $A = \bigcup_{n\geq 1} A_n$ is the set of rationals, dense in E^1. In this example we have the worst possible departure from what might have been desired, i.e., A is not only not nowhere dense, A is dense. Nevertheless, we have great things in mind for sets which can be written as countable unions of nowhere dense sets.

II.5 Definition: Let X be a topological space and let A be a subset of X. Then A is said to be *of the first category in the sense of Baire* if and only if there exists a sequence of sets $\{E_i\}_{i=1}^{\infty}$ such that each E_i is nowhere dense in X and $A = \bigcup_{i\geq 1} E_i$. A set is said to be *of the second category in the sense of Baire* if and only if it is not a set of the first category in the sense of Baire.

In the sequel we shall frequent refer to a set as a "first category set" or a "second category set," with the obvious meaning.

First category sets and second category sets are often referred to as *meager* and *non-meager*, respectively. We shall call a set *residual* if and only if it is the complement of a set of first category.

It is a curious fact that few relationships exist between the notion of category and other schemes of classifying sets. We exhibit examples of each of the following types of sets:

(i) A non-empty open set of first category.

(ii) A first category set of positive measure.

(iii) An uncountably infinite set of first category.

(iv) A dense set of first category.

(v) A first category subset of $[0,1]$ with Lebesgue measure 1.

Example (i): Let X be the set of all rational numbers with the relative topology inherited from E^1. Assume X has been enumerated: $\{r_1,r_2,\cdots\}$. For each $n \geq 1$, let $A_n = \{r_n\}$. Now each A_n is closed. Suppose $r_n = p/q$, p and q integers. Now $r_n = \lim\limits_{i\to\infty} \frac{p_i + q}{q_i}$ and each $\frac{p_i + q}{qi} \in A_n^c$. Thus II.4, (iv) tells us that each A_n is nowhere dense in X. But $X = \bigcup\limits_{n\geq 1} A_n$ and X is open in itself.

Example (ii): Let $X = [0,1]$ and let $0 < \varepsilon < 1$. Recall how, in the construction of the Cantor ternary set, one removes, in the nth stage, 2^{n-1} open intervals each of measure $(1/3)^n$. Then the measure of the set of all points taken out is 1. If at the nth stage (we are counting the stages from zero instead of from one) one removes 2^n intervals each of measure ε_n, we get a remaining set of measure $1 - \sum\limits_{n=0}^{\infty} 2^n\varepsilon_n$. Letting $\varepsilon_n = \varepsilon 2^{-2n-2}$, this gives measure $1-\varepsilon$ to the remaining set. This set is nowhere dense, as is the Cantor ternary set.

Since it is nowhere dense it is trivially of first category.

Example (iii): Since each of the Cantor-type sets described above is uncountable, our second example suffices.

Example (iv): The set of all rationals is dense in E^1 and is of first category in E^1, as was shown earlier.

Example (v): For each $n \geq 1$, let A_n be a Cantor type set of Lebesgue measure $1 - \frac{1}{n}$. Recall that each A_n is nowhere dense in $[0,1]$, so $A = \bigcup_{n\geq 1} A_n$ is of first category in $[0,1]$. Let m denote Lebesgue measure on $[0,1]$. Now $m(A) \leq 1$ since $A \subset [0,1]$. But $m(A) \geq 1 - \frac{1}{n}$ for every $n \geq 1$, since each $A_n \subset A$. Thus, $m(A) = 1$.

At this juncture it might seem that category is not a very potent concept, inasmuch as it neither implies nor is implied by some of the more familiar properties of sets in topological spaces and measure spaces. We can best dispel this idea by describing the general theme of the applications of category theory. Let X be a topological space. Suppose we wish to prove the existence of at least one element $x \in X$ which has a given property P. Since the empty set is obviously of first category in X, it would certainly suffice to show $\{x: x \in X, x \text{ has property } P\}$ is of second category. Similarly, if we already knew that X was of second category in itself, it would suffice to show that $\{x: x \in X, x \text{ does not have property } P\}$ was of first category in X. Using these ideas we shall be able to exhibit some very interesting, useful, and sometimes surprising, results.

Section 2: The Main Theorems

In this section we shall present those results which will be of the most use later. Before getting to the heart of the matter, we shall recall some topological notions which will be found useful. A topological space is a *Hausdorff space* if and only if every pair of distinct points can be separated by disjoint open sets. A space is *locally compact* if and only if every point has a neighbourhood with compact closure. A pseudo-metric is a function with all the properties of a metric except point separation, i.e., if (X,d) is a pseudo-metric space it is possible for there to exist $x \in X$ and $y \in X$, $x \neq y$, such that $d(x,y) = 0$. A pseudo-metric space is a metric space if and only if it is Hausdorff. A *regular* space is a Hausdorff space which also has the property that a closed set and a point not in that closed set can be separated by disjoint open sets. We state the following two lemmas without proof, since their proofs are standard and readily available (see, for example, Dugundji [5], p. 141 and p. 238).

II.6 Lemma: Every locally compact Hausdorff space is regular.

II.7 Lemma: A Hausdorff space X is regular if, and only if, whenever $x \in X$ and U is an open set containing x there exists an open set V such that $x \in V \subset \bar{V} \subset U$.

II.8 Definition: A topological space X is called a *Baire space* if and only if the intersection of every countable collection of open dense sets in X is dense in X.

Now we shall establish some fundamental results concerning Baire spaces, and show that some very important spaces are Baire spaces. Our first theorem is a very general form of the celebrated Baire category theorem.

II.9 Theorem: Every locally compact Hausdorff space is a Baire space.

Proof: Let X be a locally compact Hausdorff space, and let $\{D_i\}_{i=1}^{\infty}$ be a sequence of open dense sets in X. Let $D = \bigcap_{i\geq 1} D_i$. It suffices to show $D \cap U \neq \phi$, where U is an arbitrary non-empty open set. Since D_1 is dense, $D_1 \cap U \neq \phi$, and since D_1 is open, $D_1 \cap U$ is open. Since X is regular we can find a non-empty open set B_1 such that $\bar{B}_1 \subset D_1 \cap U$. Now $D_2 \cap B_1$ is non-empty and open, so we can find an open set B_2 such that $\bar{B}_2 \subset D_2 \cap B_1$. Proceeding inductively, we obtain a sequence $\{B_n\}_{n=1}^{\infty}$ such that for each $n \geq 1$, $\bar{B}_n \subset D_n \cap B_{n-1}$, and each B_n is open. Since $\bar{B}_n \subset B_{n-1}$, $\bar{B}_n \subset \bar{B}_{n-1}$, so $\{\bar{B}_n\}_{n=1}^{\infty}$ is a contracting sequence of closed sets. Since X is locally compact, we can assume $\bar{B}_1$ is compact. Thus, we have a contracting sequence of non-empty compact sets. Consequently, $B = \bigcap_{i\geq 1} \bar{B}_i$ is non-empty. Since $\bar{B}_n \subset D_n \cap U$ for each $n \geq 1$, $B \subset D_n \cap U$ for each $n \geq 1$. So $B \subset D \cap U$. But $B \neq \phi$, and thus $D \cap U \neq \phi$. This completes the proof.

We notice that the crucial factors in the above proof were the ability to put the closures of open sets inside open sets and the guarantee that the intersection of a contracting sequence of non-empty compact sets is non-empty. These factors will be crucial in proofs of more of the critical theorems.

Now that we have assumed ourselves that the class of all Baire spaces is indeed a large class, we shall derive some properties of general Baire spaces. Both of the next two theorems are generalizations of results due to Baire.

II.10 Theorem: Let X be a Baire space and let $\{A_n\}_{n=1}^{\infty}$ be a countable closed covering of X. Then at least one A_n contains a non-empty open set.

Proof: Since $X = \bigcup_{n\geq 1} A_n$, $\bigcap_{n\geq 1} A_n^c = \phi$. But since each A_n^c is open and X is a Baire space, at least one of the A_n^c is not dense, say $A_{n_o}^c$. Since $A_{n_o}^c$ is not dense, there exists a non-empty open set U such that $A_{n_o}^c \cap U = \phi$. Now $U \subset A_{n_o}$, and the proof is complete.

II.11 Definition: If X is a topological space and $A \subset X$, the interior of A is the set $A^o = \bigcup\{U: U \text{ is open, } U \subset A\}$, i.e., A^o is the largest open set contained in A.

II.12 Theorem: In a Baire space, a set of first category has empty interior.

Proof: Let X be a Baire space, and let $\{B_n\}_{n=1}^{\infty}$ be a sequence of nowhere dense sets. Let $B = \bigcup_{n\geq 1} B_n$. Suppose U is an open set, $U \subset B$. We need to show $U = \phi$. Since $B_n \subset \bar{B}_n$ for all $n \geq 1$, $\bigcup_{n\geq 1} B_n \subset \bigcup_{n\geq 1} \bar{B}_n$, so $U \subset \bigcup_{n\geq 1} \bar{B}_n$. Thus $\bigcap_{n\geq 1} (\bar{B}_n)^c \subset U^c$. Now for each $n \geq 1$, since $\bar{B}_n$ contains no open set, $(\bar{B}_n)^c$ intersects every open set and is therefore dense. Since $\bar{B}_n$ is closed, its complement is open, and since X is a Baire space we have $\bigcap_{n\geq 1} (\bar{B}_n)^c$ is dense in X. Hence, $\bar{X} = \overline{\bigcap_{n\geq 1} (\bar{B}_n)^c} \subset \overline{U^c} = U^c$ since U^c is closed. Thus, $X = U^c$ and $U = \phi$. This completes the proof.

II.13 Corollary: In a Baire space, a set of first category has dense complement.

Proof: Let X be a Baire space and let $B \subset X$ be of first category. Now $B^o = \phi$, so B contains no non-empty open set. Thus, B^c intersects every non-empty open set and is dense. This completes the proof.

When we proved our first form of the Baire category theorem, we hypothesized the Hausdorff property, but then only used it indirectly. To be sure, our proof breaks down badly if we discard this property, but it turns out that in some more restricted spaces it is unnecessary. Let (X,d) be a pseudo-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X, i.e., for every $\varepsilon > 0$ there exists a positive integer N such that $n > N$ and $m > N$ imply $d(x_n,x_m) < \varepsilon$. If every such sequence converges in X, we call (X,d) a *complete pseudo-metric space*. Note that, in general, convergent sequences in pseudo-metric spaces need not have unique limits; the Hausdorff property is needed to ensure this, and, in general, pseudo-metric spaces are not Hausdorff. Be that as it may, complete pseudo-metric spaces do turn out to be Baire spaces.

II.14 Theorem: Every complete pseudo-metric space is a Baire space.

Proof: Let (X,d) be a complete pseudo-metric space, and let $\{D_i\}_{i=1}^{\infty}$ be a sequence of open dense sets in X, with $D = \bigcap_{i\geq 1} D_i$. Now we let U be a non-empty open set in X and show $D \cap U \neq \phi$ by a method highly reminiscent of one used previously. For $x \in X$ and $r > 0$, we again write $N(x,r) = \{y: y \in X, d(x,y) < r\}$. It need not be that $\overline{N(x,r)} = \{y: y \in X, d(x,y) \leq r\}$, but we do have $\overline{N(x,r)} \subseteq \{y: y \in X, d(x,y) \leq r\}$. In particular $r_1 < r_2$ implies $\overline{N(x,r_1)} \subseteq N(x,r_2)$. Now since D_1 is open and dense, $D_1 \cap U$ is open and non-empty. Pick $x_1 \in D_1 \cap U$ and find r_1, $0 < r_1 < 1$, such that $N(x_1,2r_1) \subset D_1 \cap U$. This can be done since $D_1 \cap U$ is open. Now $\overline{N(x_1,r_1)} \subset D_1 \cap U$.

Since D_2 is open and dense, $D_2 \cap N(x_1,r_1)$ is open and non-empty. Pick $x_2 \in D_2 \cap N(x_1,r_1)$ and find r_2, $0 < r_2 < \frac{1}{2}$, such that $\overline{N(x_2,r_2)} \subset D_2 \cap N(x_1,r_1)$. Proceeding inductively, find x_n and r_n, $0 < r_n < \frac{1}{n}$, such that $\overline{N(x_n,r_n)} \subset D_n \cap N(x_{n-1},r_{n-1})$. Now this procedure generates a sequence $\{x_n\}_{n=1}^{\infty}$. Since each $N(x_n,r_n)$ is contained the previous one, x_i and x_j are in $N(x_n,r_n)$ if $i > n$ and $j > n$. Hence $d(x_i,x_j) \leq d(x_i,x_n) + d(x_n,x_j) < 2/n$, and the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. (There may be more than one, but we only need one.) Since each $x_n \in \overline{N(x_1,r_1)}$, it follows that $x \in \overline{N(x_1,r_1)}$. But $\overline{N(x_1,r_1)} \subset U$, so $x \in U$. For each $n \geq 1$, $\{x_i\}_{i=n}^{\infty} \subset \overline{N(x_n,r_n)} \subset D_n$. Since $\overline{N(x_n,r_n)}$ is closed, this says $x \in \overline{N(X_n,r_n)}$, and thus $x \in D_n$. But n was arbitrary so $x \in D_n$, for each $n \geq 1$. Thus $x \in D = \bigcap_{n\geq 1} D_n$. But we have already established that $x \in U$. Consequently, $x \in D \cap U$. In particular, $D \cap U \neq \phi$. This completes the proof.

We state the following obvious corollary without proof.

II.15 Corollary: Every complete metric space is a Baire space.

II.16 Corollary: Every complete pseudo-metric space (and therefore every complete metric space) is of the second category with respect to itself.

Proof: Let X be a complete pseudo-metric space, and assume X is of first category in itself. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of nowhere dense sets such that $X = \bigcup_{i\geq 1} E_i$. Since each E_i is nowhere dense, $\bar{E}_i$ contains no non-empty set. Hence $(\bar{E}_i)^c$ intersects every non-empty open set and is therefore dense. But each $(\bar{E}_i)^c$ is open, and X is a Baire space, so $\bigcap_{i\geq 1} (\bar{E}_i)^c$ is dense in X. Since $X = \bigcup_{i\geq 1} \bar{E}_i$, $\bigcap_{i\geq 1} (\bar{E}_i)^c = \phi$.

Since $\bar{\phi} = \phi$ and X is non-empty, this is a contradiction, so X is not of first category, i.e., X is of second category. This completes the proof.

This last result shall prove to be one of our most useful tools in the ensuing analysis.

Section 3: Some Essentially Topological Results

Before moving on to some analysis-oriented applications, we shall investigate some topological aspects of Baire spaces and some uses of category theorems in general topology. As always in topology, we shall desire to know under what conditions continuous images of Baire spaces are Baire spaces and under what conditions subspaces of Baire spaces are Baire spaces.

II.17 Theorem: Let X be a Baire space, and let Y be a topological space. Let $f: X \to Y$ be a continuous open onto mapping. Then Y is a Baire space.

Proof: Let $\{D_i\}_{i=1}^{\infty}$ be a sequence of open dense sets in Y, and let $D = \bigcap_{i \geq 1} D_i$. We need to show $D \cap V \neq \phi$ where V is an arbitrary non-empty open set in Y. For each $i \geq 1$, let $A_i = f^{-1}(D_i)$. Each A_i is open by the continuity of f. We now fix i and show A_i is dense in X. Let U be a non-empty open set in X. Now $f(U) \subset Y$ is open since f is an open mapping. Thus $f(U) \cap D_i \neq \phi$. Pick $y \in f(U) \cap D_i$. Now find x in U such that $f(x) = y$. This can be done since f is onto. Now $f(x) \in D_i$, so $x \in A_i$. But $x \in U$, so $U \cap A_i \neq \phi$, and A_i is dense in X. But i here was arbitrary, so $A = \bigcap_{i \geq 1} A_i$ is dense since X is a Baire space. Let W be a non-empty open set in Y. Now $f^{-1}(W)$ is open and is non-empty since f is onto. Thus $A \cap f^{-1}(W) \neq \phi$. Pick $x \in A \cap f^{-1}(W)$. Now $x \in f^{-1}(W)$, so $f(x) \in W$. Also $x \in A_i$ for each $i \geq 1$, so $f(x) \in D_i$ for each $i \geq 1$. Thus, $f(x) \in D$ and consequently $f(x) \in D \cap W$. Therefore $D \cap W \neq \phi$, and the proof is complete.

II.18 Theorem: An open subset of a Baire space is a Baire space.

Proof: Let X be a Baire space, A an open subset of X, and $\{D_i\}_{i=1}^{\infty}$ a sequence of open sets dense in A. Since A is open in X and each D_i is open in A, each D_i is open in X. Now, letting all closure operations be taken with respect to the parent space X, we have $A \subset \bar{D}_i$ and therefore $\bar{A} \subset \bar{D}_i$ for each i. Now let $B_i = D_i \cup (\bar{A})^c$ for each i. Clearly each B_i is open as a union of open sets. Since $\bar{B}_i = \bar{D}_i \cup \overline{(\bar{A})^c} \supset \bar{A} \cup \overline{(\bar{A})^c} \supset \bar{A} \cup (\bar{A})^c = X$ for each $i \geq 1$, each B_i is dense in X. Thus $\{B_i\}_{i=1}^{\infty}$ is a sequence of open dense sets in X. Since X is a Baire space, $B = \bigcap_{i\geq 1} B_j$ is dense in X. But $B = \bigcap_{i\geq 1} B_i = \bigcap_{i\geq 1} (D_i \cup (\bar{A})^c) = (\bigcap_{i\geq 1} D_i) \cup (\bar{A})^c = D \cup (\bar{A})^c$, where $D = \bigcap_{i\geq 1} D_i$. So $D \cup (\bar{A})^c$ is dense in X. Let U be an arbitrary non-empty open set in A. Now $(D \cup (\bar{A})^c) \cap U \neq \phi$, since U is also open in X. But $(D \cup (\bar{A})^c) \cap U = (D \cap U) \cup ((\bar{A})^c \cap U) = D \cap U$, since $U \subset A$ implies $(\bar{A})^c \cap U = \phi$. Thus, $D \cap U \neq \phi$, and D is dense in A. Hence A is a Baire space, since $\{D_i\}_{i=1}^{\infty}$ was an arbitrary sequence of open sets dense in A. This completes the proof.

Now suppose X and Y are separable and metric (we call a space *separable* if and only if it has a countable dense subset). We shall call a mapping $\phi: X \to E^1$ *upper semi-continuous* if and only if

$\{x\colon x \in X, \phi(x) < a\}$ is open in X for every real a. Let $\mathcal{a}$ be a collection of pairwise disjoint subsets of X. We call $\mathcal{a}$ an *upper semi-continuous collections* if and only if $\cup\{A\colon A \in \mathcal{a}, A \subset U\}$ is open for every open $U \subset X$. If D is a subset of X, we define the *diameter of D*, $\delta(D)$, by $\delta(D) = \sup\{d(x,y)\colon x \in D, y \in D\}$. For each positive integer k, we define a function e_k from the power set of X to the extended reals by $e_k(A) = \inf\{\max_{1\le i\le k} \delta(A_i)\}$, where $A_1,\cdots,A_k$ is a decomposition of A into pairwise disjoint non-empty subsets, and the infimum is taken over all such decompositions. If $f\colon X \to Y$, we define for $x \in X$, $e_k(x) = e_k\left(f^{-1}\{f(x)\}\right)$. This notation will be unambiguous since at all times our attention will be restricted to a particular f. Our next two results are due to G. T. Whyburn [11] and provide us with a rather pleasing view of things which can be done with Baire category theory. The first result merely lays groundwork for the second and does not use category.

II.19 Theorem: Let $f\colon X \to y$ be continuous and onto, where X and y are separable metric spaces. Suppose the decomposition of X into sets of the form $f^{-1}(\{y\})$, $y \in Y$, is upper semi-continuous. Then the function e_k, k fixed, is upper semi-continuous.

Proof: Let N be an arbitrary real number, and let $U = \{x\colon x \in X, e_k(x) < N\}$. We need to show U is open. Pick $y \in U$ and let $h = \frac{1}{3}[N - e_k(y)]$. Now $e_k(y) = N - 3h$. Since $N - 3h$ is $e_k(A)$, where $A = f^{-1}\left(\{f(y)\}\right)$, there exists a decomposition of A into k disjoint subsets $A_1,A_2,\cdots,A_k$ such that $\delta(A_i) < N\text{-}2h$ for $i = 1,\cdots,k$. Now, letting $U_r(B) = \{x\colon x \in X, d(x,B) < r\}$ for $B \subset X$, we claim $U_h(A) =$

$\bigcup_{i=1}^{k} U_h(A_i)$. If $x \in U_h(A)$, there exists $a \in A$ such that $d(a,x) < h$. But a is in some A_i, so x is in some $U_h(A_i)$ and is therefore in $\bigcup_{i=1}^{k} U_h(A_i)$. Similarly, $x \in \bigcup_{i=1}^{k} U_h(A_i)$ implies there exists i such that $x \in U_h(A_i)$, so there is $a \in A_i$ such that $d(a,x) < h$. But $a \in A_i \subset A$, so $x \in U_h(A)$. Now $\delta[U_h(A_i)] \leq \delta(A_i) + 2h < N$ for $i = 1,\cdots,k$. Let $V = \bigcup f^{-1}(\{y\})$, where the union is taken over all $y \in Y$ such that $f^{-1}(\{y\}) \subset U_h(A)$. Since we hypothesized that $\{f^{-1}(\{y\}): y \in Y\}$ is an upper semi-continuous decomposition, V is open. Also if $f^{-1}(y) \subset U_h(A)$, then $f^{-1}(y) = f^{-1}(y) \cap U_h(A) = f^{-1}(y) \cap [\bigcup_{i=1}^{k} U_h(A_i)]$ $= \bigcup_{i=1}^{k} [f^{-1}(y) \cap U_h(A_i)]$. But $\delta[f^{-1}(y) \cap U_h(A_i)] \leq \delta[U_h(A_i)] < N$ for each i. Now if $p \in V$, $f^{-1}[f(p)] \subset V$ by definition of V. So $f^{-1}(p)$ $= \bigcup_{i=1}^{k} [f^{-1}(p) \cap U_h(A_i)]$. But $\max_{1\leq j\leq k} \delta[f^{-1}(p) \cap U_h(A_i)] < N$, since each is less than N and there are only finitely many. So the greatest lower bound of all such maxima is less than N, i.e., $e_k(p) < N$. Now $x \in V \subset U_h(A)$, and since each $\delta[U_h(A_i)]$ is less than N, $e_k[U_h(A)] < N$ and $U_h(A) \subset U$. So for $x \in U$ we have found an open set V such that $x \in V \subset U$. Thus, U is open and the proof is complete.

Now if $f: X \to Y$ is onto we shall call f *k-fold* if and only if $f^{-1}(y)$ contains at least k points, for each $y \in Y$. We shall call f irreducibly k-fold if and only if f is k-fold and whenever $D \subset X$ is a proper closed subset then there exists $y \in Y$ such that $D \cap f^{-1}(y)$ has less than k elements. Our next theorem is Whyburn's major result in [11], and we shall find that it relies heavily on category theory. We recall that a compact metric space is complete.

II.20 Theorem: Let X be a compact separable metric space and let Y be a separable metric space. Also let $f: X \to Y$ be continuous and onto. Then f is irreducibly k-fold if and only if $\{x: x \in X, f^{-1}[f(x)]$ has exactly k points$\}$ is dense in X.

Proof: Let $D = \{x: x \in X, f^{-1}[f(x)]$ has exactly k points$\}$. Suppose D is dense in X. Let $A \subset X$ be an arbitrary proper closed subset. Now A^c is a non-empty open set, so $D \cap A^c \neq \phi$, i.e., there exists $x \in A^c$ such that $f^{-1}[f(x)]$ has exactly k points. But since $x \in f^{-1}[f(x)]$, this says $f^{-1}[f(x)]$ has at most k-1 points in A. But A was an arbitrary proper closed subset of X, so f is irreducibly k-fold on X.

Now suppose f is irreducibly k-fold on X. Let U be an open set, $U \subset X$, and let $U(f) = \{x: x \in X, f^{-1}(f(x)) \subset U\}$. By the previous theorem, if we can show U(f) is open we will have shown e_k is upper semi-continuous. We shall show $X - U(f)$ is closed. Let x be a limit point of $X - U(f)$, and let $\{x_i\}_{i=1}^{\infty} \subset X - U(f)$ converge to x. For each $i \geq 1$ find $y_i \in f^{-1}(f(x_i))$ such that $y_i \notin U$. Now U^c is a closed subset of compact X, so $\{y_i\}_{i=1}^{\infty}$ has a convergent subsequence $\{y_{i_n}\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_{i_n} = y \notin U$. Now $f(y_{i_n}) = f(x_{i_n})$ for each $n \geq 1$, so $f(y) = f(x)$ and $y \in f^{-1}(f(x))$. Thus $x \in X - U(f)$ since $y \notin U$. So U(f) is open and e_k is upper semi-continuous.

For each positive integer n, let $V_n = \{x: x \in X, e_k(x) < \frac{1}{n}\}$. Now each V_n is open by the upper semi-continuity of e_k. Fix n and let U be a non-empty open set such that $\delta(U) < \frac{1}{n}$. Now U^c is a proper closed subset of X, so there is $y \in Y$ such that U^c contains at most k-1 points of $f^{-1}(y)$. Thus, $f^{-1}(y)$ is contained in U united with at most k-1 additional points. So $f^{-1}(y)$ has a decomposition into k sets each of diameter less than 1/n. So $x \in f^{-1}(y)$ implies $e_k(x) < 1/n$. But $U \cap f^{-1}(y) \neq \phi$, so there is $x \in U$ such that $e_k(x) < 1/n$. Thus, $V_n \cap U \neq \phi$ for every non-empty open set of diameter less than 1/n. If an open set U has $\delta(U) \geq 1/n$, U certainly contains an open set of diameter less than 1/n, so $V_n \cap U \neq \phi$ for every non-empty open set U   X. So each V_n is dense in X. Thus $\{V_n\}_{n=1}^{\infty}$ is a sequence of open dense sets in X, a complete metric space. Consequently, $V = \bigcap_{i \geq 1} V_i$ is dense, by II.15. But $V = \bigcap_{i \geq 1} \{x: x \in X, e_k(x) < 1/i\} = \{x: x \in X, e_k(x) = 0\}$, and the zeros of e_k are dense in X. But $e_k(x) = 0$ implies $f^{-1}(f(x))$ has $\leq k$ elements. Since f is k-fold, every $f^{-1}(f(x))$ has $\geq k$ elements, so V = D, and D is dense in X. This completes the proof.

Now that we have demonstrated the power of category in topology, we shall move on and devote Chapter III to results from classical real analysis.

CHAPTER III

APPLICATIONS TO CLASSICAL ANALYSIS

Section 1: Some Particular Complete Metric Spaces

We shall now assume, with no further comment, that the set of all real numbers with the usual topology, denoted E^1, is a complete metric space, i.e., every Cauchy sequence of real numbers has a unique limit. Frequently we shall be concerned more with closed subsets of E^1 than with E^1 itself, so we shall prove the following useful result.

III.1 Theorem: Every closed subset of a complete metric space is complete.

Proof: Let (X,d) be a complete metric space with D a closed subset of X. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in D. Since $D \subset X$, $\{x_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in X, and thus there exists $x \in X$ such that $x = \lim\limits_{n\to\infty} x_n$. Now, given $\varepsilon > 0$, there exists a positive integer N such that $n > N$ implies $d(x_n,x) < \varepsilon$. Since each x_n is in D, this says that every ε-neighbourhood of x intersects D, in which case $x \in \bar{D}$. But D is closed, so $x \in D$. Since $\{x_n\}_{n=1}^{\infty}$ was an arbitrary Cauchy sequence in D, this asserts that D, considered as a space, is complete. This completes the proof.

This theorem tells us in particular that every closed interval $[a,b]$, $b \geq a$, is a complete metric space. Now for each $[a,b]$, we let $C[a,b]$ denote the set of all continuous real-valued functions on $[a,b]$. Now fix a and b, and for $f \in C[a,b]$ and $g \in C[a,b]$, let

$d(f,g) = \max\{|f(x) - g(x)|: x \in [a,b]\}$. Since $[a,b]$ is compact and $|f-g|$ is continuous whenever f and g are, $d(f,g)$ is well-defined for f and g in $C[a,b]$. Since $d(f,g)$ is the maximum of a set of non-negative real numbers, clearly $d(f,g) \geq 0$ for any f and g in $C[a,b]$. If $d(f,g) = 0$, then $|f(x) - g(x)| = 0$ for all $x \in [a,b]$ and hence $f(x) = g(x)$ for all x in $[a,b]$. Thus, $d(f,g) = 0$ implies $f = g$. Since $|f(x) - g(x)| = |g(x) - f(x)|$ for every $x \in [a,b]$, $d(f,g) = d(g,f)$. Now let f,g, and h be in $C[a,b]$. Then

$$\begin{aligned}
d(f,g) &= \max\{|f(x) - g(x)|: a \leq x \leq b\} \\
&= \max\{|f(x) - h(x) + h(x) - g(x)|: a \leq x \leq b\} \\
&\leq \max\{|f(x) - h(x)| + |h(x) - g(x)|: a \leq x \leq b\} \\
&\leq \max\{|f(x) - h(x)|: a \leq x \leq b\} + \\
&\qquad\qquad \max\{|h(x) - g(x)|: a \leq x \leq b\} \\
&= d(f,h) + d(h,g).
\end{aligned}$$

This last inequality, called the triangle inequality, along with the previous remarks, shows that d is metric, called the uniform metric, and that $(C[a,b],d)$ is a metric space. Hereafter we shall denote $(C[a,b],d)$ simply by $C[a,b]$ and assume the metric to be understood. Now we shall show that this metric space is complete.

III.2 Theorem: Let a and b be real, $b > a$. Then $C[a,b]$ is a complete metric space.

Proof: Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C[a,b]$. Given $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ and $m \geq N$ implies

$$|f_n(x) - f_m(x)| \leq d(f_n,f_m) < \varepsilon$$

for each $x \in [a,b]$, so $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers for each x in $[a,b]$. Define f on $[a,b]$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Now we show f is continuous. Let $\varepsilon > 0$ be given and fix $x \in [a,b]$. Find N so that $n \geq N$ and $m \geq N$ implies $d(f_n,f_m) < \varepsilon/5$. Now find $\delta > 0$ such that $|x-y| < \delta$ implies $|f_N(x) - f_N(y)| < \varepsilon/5$. This can be done since f_N is continuous. Now let y be such that $|x-y| < \delta$ and find $n \geq N$ and $m \geq N$ such that $|f(x) - f_n(x)| < \varepsilon/5$ and $|f(y) - f_m(y)| < \varepsilon/5$. Now

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_N(x) + f_N(x) - f_N(y) \\ &\qquad + f_N(y) - f_m(y) + f_m(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| \end{aligned}$$

$$+ |f_N(y) - f_m(y)| + |f_m(y) - f(y)|$$

$$< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon.$$

Thus, f is continuous on $[a,b]$ since $x \in [a,b]$ was arbitrary.

We now show $d(f_n,f) \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$ and find N such that $n \geq N$ and $m \geq N$ implies $d(f_n,f_m) < \varepsilon/2$. Let $n \geq N$. Now for $x \in [a,b]$,

$$|f(x) - f_n(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)|.$$

Choose m so large that $m \geq N$ and that $|f(x) - f_m(x)| < \varepsilon/2$ (note that although m may depend on x, n does not). With m so chosen, we have $|f(x) - f_n(x)| < \varepsilon$. But x was arbitrary, so $d(f,f_n) < \varepsilon$ for $n > N$, and the proof is complete.

Armed with these results, we shall in Section 2 exploit the completeness of the reals and closed subsets of the reals, and in Section 3 we shall use the completeness of $C[a,b]$.

Section 2: Some Properties of Repeated Integration and Repeated Differentiation

Suppose f is a continuous real-valued function on $[0,1]$. Let f_1 be any antiderivative of f, i.e., f_1 satisfies $f_1' = f$ on $[0,1]$. Proceeding inductively, assume that $f_1,\cdots,f_k$ have been chosen, where k is a positive integer. Then let f_{k+1} be any function on $[0,1]$ such that $f_{k+1}' = f_k$ on $[0,1]$. Now by this scheme we have generated a sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on $[0,1]$. Now suppose that some f_k

were identically zero on [0,1]. Then by differentiating k consecutive times it follows that f is identically zero on [0,1]. It turns out that with much less information than that some f_k is identically zero, we can still assert that f is the zero function on [0,1].

III.3 Theorem: Suppose f and $\{f_n\}_{n=1}^{\infty}$ are as described above, and suppose furthermore that for each $x \in [0,1]$ there is at least one index k such that $f_k(x) = 0$. Then $f(x) = 0$ for every x in [0,1].

Proof: For each integer k, let $E_k = \{x: 0 \leq x \leq 1, f_k(x) = 0\}$. Note that each E_k is closed. By hypothesis, $[0,1] = \bigcup_{k\geq 1} E_k$. Since [0,1] is a complete metric space, and therefore a Baire space, this asserts that there is at least one E_k which is not nowhere dense, i.e., $\bar{E}_k$ contains a non-empty open set, say U_k. Now f_k vanishes on E_k and, since f_k is continuous, it also vanishes on $\bar{E}_k$ and therefore on U_k. But U_k is open, so repeated differentiation gives $f(x) = 0$ for all $x \in U_k$.

Now let $x_o \in [0,1]$ be arbitrary and suppose there exists $\varepsilon > 0$ such that f is never zero in $N(x_o,\varepsilon) \cap [0,1]$. Let $D = \{x: 0 \leq x \leq 1, |x-x_o| \leq \varepsilon/2\}$. Now D is a closed subset of $N(x_o,\varepsilon) \cap [0,1]$, so D is a complete metric space. Since $D = \bigcup_{k\geq 1} (D \cap E_k)$, at least one set $D \cap E_k$ is not nowhere dense in D. Thus, by the contruction used above, we can find an open set $V \subset D$ so that $f(x) = 0$ for every $x \in V$. This is a contradiction, and consequently the set

$$\{x: 0 \leq x \leq 1, f(x) = 0\}$$

intersects every non-empty open subset of [0,1], i.e., is dense in

[0,1]. But since f is continuous,

$$\{x\colon 0 \le x \le 1,\ f(x) = 0\}$$

is a closed set and hence equals [0,1]. This completes the proof.

The following theorem is both more interesting and more intricate in its proof than was the last. Let us agree to call a subset of E^1 *perfect* if, and only if, it is closed and each of its points is a limit point, i.e., a perfect set is a closed set with no isolated points. Now again let f be a continuous real-valued function on [0,1], but suppose in addition that f has all its derivatives on [0,1]. If some derivative, say $f^{(n)}$, were identically zero on [0,1], then straightforward application of the Mean Value Theorem would tell us that f is a polynomial. This result generalizes in a fashion analogous to the last result.

III.4 Theorem: Let f be an infinitely differentiable real-valued function [0,1], and suppose that for each $x \in [0,1]$ there is a positive integer n such that $f^{(n)}(x) = 0$. Then f is a polynomial on [0,1].

Proof: Throughout this proof, sets will be considered open or closed relative to [0,1]. For each n, let

$$E_n = \{x\colon 0 \le x \le 1,\ f^n(x) = 0\}.$$

Now $[0,1] = \bigcup\limits_{n\ge 1} E_n$, so from Baire's theorem we have that at least one E_n contains a non-empty open set, and therefore a non-empty open interval, throughout which $f^{(n)}$ is zero. On this open interval, f is a polynomial. Let P be the union of all open intervals throughout which f is a polynomial, i.e.,

$$P = \bigcup\{(a,b)\colon \text{f is a polynomial on } (a,b)\}.$$

Our above remarks show $P \neq \phi$. We now show P is dense in $[0,1]$. Pick $x \in [0,1]$ and $\varepsilon > 0$. Now $[x - \varepsilon/2, x + \varepsilon/2]$ is a complete metric space, so the above argument applies, and $P \cap [x - \varepsilon/2, x + \varepsilon/2] \subset P \cap (x-\varepsilon, x+\varepsilon) \neq \phi$. Thus P is dense in $[0,1]$.

Let $H = P^c$. Suppose $H = \phi$. Then $[0,1]$ is covered by open sets throughout each of which f is a polynomial. By compactness of $[0,1]$, we can find a finite subcover. If n is a positive integer greater than any of the degrees of the polynomials associated with the open intervals of the subcover, $f^{(n)}(x) = 0$ for $x \in [0,1]$, so f is a polynomial.

Assume $H \neq \phi$. Since P is open, H is closed. Now we wish to show H is perfect, i.e., H has no isolated points. Suppose $y \in H$ is isolated. Find c and d, $c < d$, so that $y \in [c,d]$, and $[c,d] \cap H = \{y\}$. Choose y' such that $c < y' < y$. Now $[c,y'] \subset P$, and is compact, so we can find a finite subcover as we did in the last paragraph, and we have that f is a polynomial on $[c,y']$. Let $\{y_o = y', y_1, y_2, \cdots\}$ be a monotonic sequence approaching y from the left. Now f is a polynomial on each $[c,y_n]$. But we have the same polynomial every time, since $[c,y'] \subset [c,y_n]$ for every n, and two polynomials agreeing on an infinite set are identical. Thus, f is a polynomial on $[c,y)$, say of degree s. Similarly, f is a polynomial on $(y,d]$, say of degree t. Let

$k > \max\{s,t\}$. Now $f^{(k)}(x) = 0$ for $c \leq x < y$ and for $y < x \leq d$. But $f^{(k)}$ is continuous, so $f^{(k)}(y) = 0$. Thus, $f^{(k)}$ is zero on $[c,d]$ and hence on (c,d), so $(c,d) \subset P$, and $y \notin H$. This is a contradiction, and H is perfect.

We continue to assume that H is non-empty. Since H is a closed subset of a complete metric space, H is a complete metric space. For each positive integer n, let $E_n = \{x\colon 0 \leq x \leq 1,\ f^{(n)}(x) = 0\}$. Now by Baire's theorem, at least one E_n is dense throughout an interval of H, i.e., a set of the form $J \cap H$ where J is an interval of $[0,1]$. Hence, $f^{(n)}(x) = 0$ for each x in $J \cap H$. Since H^c is dense, J also contains some intervals of H^c (if this were not so, H would have to have an isolated point in J, which is impossible since H is perfect). Let K be an open interval of $J \cap H^c$. Now there exists an index m such that $f^{(m)}(x) = 0$ for all x in K. Suppose $m \leq n$. Then by differentiating $f^{(m)}$ n-m times, we find $f^{(n)}$ to be zero throughout K. Now suppose $m > n$. Let $x \in J \cap H$. Since H is perfect, x is a limit point of H, so for each integer k there exists a point $y_k \in J \cap H$ such that $|x - y_k| < 1/k$. Now

$$\frac{f^{(n)}(x) - f^{(n)}(y_k)}{x - y_k} = 0$$

for every k. Since $y_k \to x$ as $k \to \infty$, and since $f^{(n)}$ is known to be differentiable at x, this gives $f^{(n+1)}(x) = 0$. But $x \in J \cap H$ was arbitrary. Hence, $f^{(n+1)}(x) = 0$ for every x in $J \cap H$. Similarly, $f^{(n+2)}(x) = f^{(n+3)}(x) = \cdots = 0$ for every $x \in J \cap H$. Let a be the left endpoint of K, and note that $a \in J \cap H$. Now for x in K,

$$f^{(n)}(x) = f^{n}(a) + (x-a)f^{(n+1)}(a) + \frac{(x-a)^2}{2!} f^{(n+2)}(a)$$

$$+ \cdots + \frac{(x-a)^{m-n-1}}{(m-n-1)!} f^{(m-1)}(a) + \frac{1}{(m-n-1)!} \int_a^x (x-t)^{m-n-1} f^{(m)}(t)dt,$$

by Taylor's theorem. Since $a \in J \cap H$, $f^{(n)}(a) = f^{(n+1)}(a) = \cdots = f^{(m-1)}(a) = 0$. Since $x \in K$, $f^{(m)}(t) = 0$ on (a,x). Thus $f^{(n)}(x) = 0$ for x in K. We have shown that $f^{(n)}(x) = 0$ everywhere in J, which implies $J \cap H = \phi$. By assuming $J \cap H \neq \phi$, we deduced $J \cap H = \phi$, a contradiction. Consequently $J \quad H = \phi$. But this contradicts Baire's theorem *unless* $H = \phi$. Thus $H = \phi$, and there is only one U_k. This completes the proof. Since intuition doesn't assign much meaning to the zeros of derivatives of any higher order than two, this last result cannot be considered particularly intuition defying. Some results which are more intuition defying can be obtained by applying Baire's theorem to the complete metric space $C[a,b]$.

Section 3: Some Results Concerning Continuous Functions Considered as Elements of C[a,b]

Recall that if $f \in C[0,1]$ and $\varepsilon > 0$, then the ε-neighbourhood of f consists of all continuous functions g on $[0,1]$ such that $f(x) - \varepsilon < g(x) < f(x) + \varepsilon$ for every $x \in [0,1]$. Now suppose (a,b) is an open interval in E^1 with $a < b$. Since $a < b$, we can find two national numbers, r_1 and r_2, such that $a < r_1 < r_2 < b$ and therefore $(r_1,r_2) \subset (a,b)$. Thus, every open interval contains an open interval with rational endpoints. Thus, if a continuous function f is monotonic in some open interval (a,b), then f is also monotonic in an interval with rational endpoints. This implies that the set of all continuous functions which are monotonic in any open interval is the same as the set of all continuous functions which are monotonic in some interval with rational endpoints.

III.5 Theorem: There exists a continuous function f on [0,1] such that f is monotonic in no open interval in [0,1], i.e., f is *everywhere oscillating* on [0,1].

Proof: As has been outlined above, it suffices to show that there exists a continuous function monotonic in no open interval with rational endpoints. Since the set of all open intervals in [0,1] with rational endpoints is countable, assume that this set of intervals is enumerated in the sequence $\{I_1, I_2, \cdots, I_n, \cdots\}$. For each positive integer n, let

$$E_n = \{f\colon f \in C[0,1],\ f \text{ is monotonic in } I_n\}.$$

Now if we show that each set E_n is nowhere dense in C[0,1], then, since C[0,1] is a complete metric space, $\bigcup_{n\geq 1} E_n$ cannot be all of C[0,1]. So there will exist at least one f $\in$ C[0,1] which is in no E_n, i.e., such that f is monotonic in no I_n. To show E_n is nowhere dense, we show E_n^c, the complement of E_n, is both open and dense in C[0,1].

First, we show E_n^c is open. Suppose $f \in E_n^c$. Then f is not monotonic in I_n. Thus, we can find points x, y, and z in I_n, with $x < y < z$, such that either $f(x) < f(y)$ and $f(z) < f(y)$ or $f(x) > f(y)$

and $f(z) > f(y)$. With no loss of generality we assume $f(x) < f(y)$ and $f(z) < f(y)$. Now choose ε such that

$$0 < \varepsilon < \min \{1/2[f(y)-f(x)], \quad 1/2[f(y)-f(z)]\}.$$

Suppose that $g \in C[0,1]$ with $d(f,g) < \varepsilon$. Then $g(y) > f(y) - \varepsilon$. Since $f(y) - f(x) > 2\varepsilon$, $f(y) > f(x) + 2\varepsilon$. Thus

$$f(y) - \varepsilon > f(x) + \varepsilon > g(x),$$

and consequently $g(y) > g(x)$. A similar argument shows that $g(y) > g(z)$. Thus $g \in E_n^c$ and hence $N(f,\varepsilon)$ E_n^c. Consequently, the set E_n^c is open in $C[0,1]$.

Now to show that E_n^c is dense we show that given $f \in C[0,1]$ and $\varepsilon > 0$, there exists $g \in E_n^c$ such that $d(f,g) < \varepsilon$. We shall merely outline a proof here, without all the cumbersome details. First, we consider the strip

$$\{(x,y)\colon 0 \le x \le 1,\ f(x) - \varepsilon/2 < y < f(x) + \varepsilon/2\}$$

in E^2. In this strip we can, on the basis of the uniform continuity of f on $[0,1]$, place a polygonal line, i.e., we can find a piecewise linear function $g_1 \in N(f,\varepsilon/2)$. Thus, the "slope" of g_1 is bounded. Now let g_2 be a sawtooth function such that $|g_2(x)| < \varepsilon/2$ for every $x \in [0,1]$. Given I_n, we can make the "teeth" of g_2 sufficiently steep that $g_1 + g_2$ is not monotonic in I_n, in which case $g_1 + g_2 \in E_n^c$.

Thus, E_n^c is dense in $C[0,1]$, and therefore E_n is nowhere dense. This completes the proof, since the complement of an open dense set is indeed nowhere dense.

Although it is well known that continuity of a real-valued function need not imply its differentiability, it may seem on intuitive grounds that a continuous function is differentiable in "most" places. That there do indeed exist continuous nowhere differentiable functions was shown by Weierstrass in the nineteenth century ([6], p. 258). Weierstrass's existence proof was an existence proof of the most convincing kind, a construction. But the exhibition of a particular example does not entirely clarify the general phenomenon involved in this problem. Now if a continuous function has a finite derivative at a particular point, it certainly has a finite right-hand derivative at that point. To give some notion of "how many" continuous nowhere differentiable functions there are, we shall show that the set of all $f \in C[0,1]$ which have a finite right-hand derivative at even one point in $[0,1]$ is of first category in $C[0,1]$.

III.6 Theorem: If F is the subset of $C[0,1]$ composed of those functions which have a finite right-hand derivative at at least one point of $[0,1]$, then F is of first category in $C[0,1]$.

Proof: Suppose $x \in [0,1]$, and f has a finite right-hand derivative at x. Since this notion has no meaning at $x = 1$, there will be an integer N_1 such that

$$x \in [0,1 - 1/N_1] \subset [0,1 - 1/n]$$

for $n \geq N_1$. Now since

$$\lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

is finite, there is an integer N_2 such that

$$\left|\lim_{h\to 0^+} \left[\frac{f(x+h) - f(x)}{h}\right]\right| < N_2.$$

Consequently, there exists $\varepsilon > 0$ such that

$$\left|\frac{f(x+h) - f(x)}{h}\right| < N_2$$

if $0 < h < \varepsilon$. Let N_3 be a positive integer such that $1/N_3 < \varepsilon$. Now let $N = \max\{N_1, N_2, N_3\}$. The above remarks show that

$$x \in [0, 1 - 1/N],$$

and that

$$\left|\frac{f(x+h) - f(x)}{h}\right| < N$$

whenever $0 < h < 1/N$. For each positive integer n, let

$$E_n = \{f\colon f \in C[0,1], \text{ there exists } x \in [0,1-1/n]$$

such that $0 < h < 1/n$ implies

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq n\}.$$

The preceding remarks show that every $f \varepsilon C[0,1]$ which has a finite right-hand derivative at any point is in some E_n. Hence, $F \subset \bigcup_{n\geq 1} E_n$. Thus, to complete the proof of the theorem, it suffices to show that each set E_n is nowhere dense in $C[0,1]$. As in the last theorem, we shall do this by showing E_n is closed and E_n^c is dense.

First we show E_n is closed. Let $\{f_k\}_{k=1}^{\infty}$ be a convergent sequence in E_n with limit f. We need to show $f \varepsilon E_n$. For each k find $x_k \varepsilon [0,1-1/n]$ such that $0 < h < 1/n$ implies

$$\left|\frac{f_k(x_k+h) - f_k(x_k)}{h}\right| \leq n.$$

Now $\{x_1,x_2,\cdots\}$ is an infinite set in $[0,1-1/n]$, a compact set. Thus $\{x_k\}_{k=1}^{\infty}$ has a convergent subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ with limit $x_o \varepsilon [0,1-1/n]$. Fix h, $0 < h < 1/n$, and let $\varepsilon > 0$ be given. Find $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \frac{h\varepsilon}{4}$ (this can be done since $f \varepsilon C[0,1]$ is uniformly continuous on $[0,1]$). Let N_1 be a positive integer such that $i > N_1$ implies $|x_{k_i}-x_o| < \delta$. Now find N_2 such that $i > N_2$ implies $d(f,f_i) < \frac{h\varepsilon}{4}$. Now $i > \max\{N_1,N_2\}$ implies

$$\left|\frac{f(x_o+h) - f(x_o)}{h}\right| < \left|\frac{f(x_{k_i}+h) - f(x_{k_i})}{h}\right| + \varepsilon/2$$

$$< \left| \frac{f_i(x_{k_i}+h) - f_i(x_{k_i})}{h} \right| + \varepsilon \le n + \varepsilon.$$

So

$$\left| \frac{f(x_o+h) - f(x_o)}{h} \right| < n + \varepsilon$$

for every $\varepsilon > 0$. Thus

$$\left| \frac{f(x_o+h) - f(x_o)}{h} \right| \le n.$$

But $h \in (0,1/n)$ was arbitrary so this holds for every $h \in (0,1/n)$. Consequently, $f \in E_n$ and E_n is closed.

Now let $f \in C[0,1]$ be arbitrary, and let $\varepsilon > 0$ be given. Let g_1 be a polygonal function on $[0,1]$ such that $d(f,g_1) < \varepsilon/2$. Now g_1, being polygonal, has right derivatives at all points of $[0,1)$, and furthermore the set of values of its right derivative is finite, and therefore bounded. Let $M > 0$ be a bound on the value of the right derivative of g_1. Let n be an arbitrary positive integer. Find an integer $k > n + M$. Let j be a positive integer with $k/j < \varepsilon/4$. Now define g_2 on $[0,2/j]$ by

$$g_2(x) = \begin{cases} kx, & 0 \le x \le 1/j \\ -kx + 2k/j, & 1/j \le x \le 2/j. \end{cases}$$

If we now extend g_2 periodically, we get $|g_2(x)| < \varepsilon/2$ for all $x \in [0,1]$, and g_2 has a right derivative with constant absolute value k. Thus, $g_1 + g_2 \in N(f,\varepsilon)$ and $g_1 + g_2$ has right derivative always greater in absolute value than n, i.e., $g_1 + g_2 \in E_n^c$. Thus, E_n^c is dense, and the proof is complete.

CHAPTER IV

BAIRE CATEGORY THEORY IN FUNCTIONAL ANALYSIS

Section 1: Some Preliminary Results from Functional Analysis

Throughout this chapter we shall be concerned chiefly with normed linear spaces in general, and Banach spaces in particular. First we need to define these and several related notions.

IV.1 Definition: A *complex linear space* X is a vector space over the field of complex numbers. A *real linear space* X is a vector space over the field of real numbers.

Unless stated otherwise, all linear spaces in what follows shall be assumed to be complex linear spaces.

IV.2 Definition: A linear space X is a *normed linear space* if to each $x \in X$ there is assigned a non-negative real number $\|x\|$, called the *norm* of x, such that

(i) $\|x+y\| \leq \|x\| + \|y\|$ for all x and y in X

(ii) $\|\alpha x\| = |\alpha| \; \|x\|$ for all $x \in X$ and α a scalar (complex number).

(iii) $\|x\| = 0$ if and only if $x = 0$.

If x, y and z are in X, it follows from (i) that

$$\|x-z\| = \|(x-y) + (y-z)\| \leq \|x-y\| + \|y-z\| .$$

Note that $\|x-y\| = 0$ if and only if $x = y$, and

$\|x-y\| = \|(-1)(y-x)\| = |-1|\,\|y-x\| = \|y-x\|$. Thus, if we define $d(x,y) = \|x-y\|$, (X,d) is a metric space. This metric, d, is called the *metric induced by the norm.* If a normed linear space is complete in the metric induced by its norm, the space is called a Banach space.

A transformation T from a normed linear space X into a normed linear space Y is said to be *linear* if and only if whenever x and y are in X and α and β are scalars,

$$T(\alpha x + \beta y) = \alpha\, Tx + \beta\, Ty.$$

If $T\colon X \to Y$ is a linear transformation, we define the *norm of T,* written $\|T\|$, by

$$\|T\| = \sup \left[\frac{\|Tx\|}{\|x\|} : x \in X,\ x \neq 0\right].$$

Note that in a typical expression of the form $\|Tx\|/\|x\|$ the norm in the numerator is that of the space Y, and the norm in the denominator is that of the space X. Such a mixture of notations is perhaps unfortunate, but it is standard and the context is usually such that any confusion is unlikely. If $\|T\| < \infty$, T is called a *bounded linear transformation.* Since every number of the form $\|Tx\|/\|x\|$, $x \neq 0$, can be rewritten

$$\frac{\left\| T\left[\frac{x}{\|x\|}\right] \right\|}{\left\| \frac{x}{\|x\|} \right\|} = \left\| T\left[\frac{x}{\|x\|}\right] \right\| ,$$

we have

$$\|T\| = \sup \{\|Tx\| : x \in X, \|x\| = 1\}.$$

Note also that $\|T\|$ is the smallest number such that $\|Tx\| \leq \|T\| \, \|x\|$ for all $x \in X$. This last inequality proves to be extremely useful.

Next we shall prove a theorem which has far-reaching consequences. It turns out that in normed linear spaces, the boundedness of a linear transformation implies its continuity, and its continuity implies its boundedness.

IV.3 Theorem: Let T be a linear transformation from a normed linear space X into a normed linear space Y. Then the following three conditions are equivalent:

(i) T is bounded,

(ii) T is continuous,

(iii) T is continuous at one point of X.

Proof: If T is the zero transformation, the theorem is obvious as all three conditions are fulfilled. Suppose T is not the zero transformation.

We prove (i) implies (ii). Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that $\delta < \varepsilon/\|T\|$. Now $\|x_1 - x_2\| < \delta$ implies

$$\|Tx_1-Tx_2\| = \|T(x_1-x_2)\|$$

$$\leq \|T\| \ \|x_1-x_2\|$$

$$< \|T\| \ \delta < \varepsilon.$$

Thus, T is continuous.

That (ii) implies (iii) is obvious.

We now show (iii) implies (i). Suppose T is continuous at $x_o \varepsilon X$. Fix $\varepsilon > 0$ and find $\delta > 0$ such that $\|x-x_o\| < \delta$ implies $\|Tx-Tx_o\| < \varepsilon$. If $x \varepsilon X$, $\|x\| < \delta$, then $\|(x_o+x) - x_o\| < \delta$, so $\|T(x_o+x) - Tx_o\| < \varepsilon$. But $T(x_o+x) - Tx_o = Tx$, so $\|Tx\| < \varepsilon$. Now suppose $\|x\| = 1$. Then $\|\delta/2\ x\| = \delta/2 < \delta$. Hence,

$$\frac{\|Tx\|}{\|x\|} = \frac{\|T(\delta/2\ x)\|}{\|\delta/2\ x\|} = 2/\delta\|T(\delta/2\ x)\| < 2\varepsilon/\delta.$$

Consequently, $\sup \{\|Tx\|: x \varepsilon X, \|x\| = 1\} \leq 2\varepsilon/\delta < \infty$, and T is bounded. This completes the proof.

Section 2: On a Uniform Bound for a Family of Bounded Linear Transformations

We are now prepared to prove the Banach-Steinhaus theorem. This theorem is also known as the *uniform boundedness theorem.*

IV.4 Theorem (Banach-Steinhaus): Suppose X is a Banach space, Y is a normed linear space, and $\{T_a: a \varepsilon A\}$ is a collection of bounded linear transformations from X to Y, where A is a non-empty indexing set.

Then one of the following must be true:

(i) There exists a real number $M > 0$ such that $\|T_a\| \leq M$ for every $a \varepsilon A$.

(ii) $\sup \{\|T_a x\|: a \varepsilon A\} = \infty$ for all x in a dense G_δ set of the space X.

Proof: First note that the triangle inequality tells us the norm on Y is a continuous function from Y to E^1. Now for each $x \varepsilon X$, let

$$\phi(x) = \sup \{\|T_a x\|: a \varepsilon A\}.$$

We acknowledge the possibility of ϕ being extended real-valued instead of simply real-valued. For each positive integer n, let $V_n = \{x: x \varepsilon X, \phi(x) > n\}$. Now for arbitrary n we claim

$$V_n = \bigcup_{a \varepsilon A} \{x: \|T_a x\| > n\}.$$

If $x \varepsilon \bigcup_{a \varepsilon A} \{x: \|T_a x\| > n\}$, there exists $b \varepsilon A$ such that $\|T_b x\| > n$. But $\phi(x) \geq \|T_b x\|$, so $\phi(x) > n$ and $x \varepsilon V_n$. Now suppose $x \varepsilon V_n$. Then

$$\sup \{\|T_a x\|: a \varepsilon A\} > n.$$

If $\|T_a x\| \leq n$ for each $a \varepsilon A$, we would have

$$\sup \{\|T_a x\|: a \varepsilon A\} \leq n.$$

Thus, there exists $b \varepsilon A$ such that $\|T_b x\| > n$. Consequently,

$$x \varepsilon \{x: \|T_b(x)\| > n\} \subset \bigcup_{a \varepsilon A} \{x: \|T_a x\| > n\}.$$

But since the norm on Y is continuous, each $\{x: \|T_a x\| > n\}$ is open by the continuity of T_a. Thus, each V_n is a union of open sets and is therefore open. Now either every V_n is dense in X or at least one V_N is not dense in X.

Suppose there exists a positive integer N such that V_n is not dense in X. Then there is an open set in X which V_N does not intersect, and therefore there exists $x_o \varepsilon X$ and $r > 0$ such that $\|x\| < r$ implies $x_o + x \notin V_N$. So $\phi(x_o+x) \leq N$ if $\|x\| < r$, i.e., $\|T_a(x_o+x)\| \leq N$ for every $a \varepsilon A$ if $\|x\| < r$. Since $x = (x_o+x) - x_o$, we have

$$\|T_a x\| = \|T_a(x_o+x) - T_a x_o\| \leq \|T_a(x_o+x)\| + \|T_a x_o\| \leq 2N.$$

Now suppose $x \varepsilon X$, $\|x\| = 1$. Then

$$\frac{1}{2} r \|T_a x\| = \|T_a(1/2\ rx)\| \leq 2N,$$

and $\|T_a x\| \leq \frac{4N}{r}$ for every $x \varepsilon X$ such that $\|x\| = 1$. Thus $\|T_a\| \leq \frac{4N}{r}$. But a here is arbitrary, so, setting $M = \frac{4N}{r}$, we have proved (i).

Now suppose every V_n is dense in X. Since each V_n is open and X is a complete metric space, we have that $V = \bigcup_{n \geq 1} V_n$ is dense in X. But $x \varepsilon V$ implies $\phi(x) = \infty$. Consequently, (ii) is proved since V is a dense G_δ. This completes the proof.

An immediate corollary of the Banach-Steinhaus theorem is Banach's resonance theorem.

IV.5 Theorem: Suppose X is a Banach space, Y is a normed linear space, and $\{T_a: a \in A\}$ is a collection of bounded linear transformations from X to Y. Suppose the set

$$\{\|T_a x\|: a \in A\}$$

is bounded for each $x \in X$. Then

$$\{\|T_a\|: a \in A\}$$

is bounded.

Proof: Since $\{\|T_a x\|: a \in A\}$ is bounded for each $x \in X$, $\sup \{\|T_a x\|: a \in A\} < \infty$ for each $x \in X$. Since this rules out the second alternative of the Banach-Steinhaus theorem, we deduce that the first alternative applies, and $\{\|T_a\|: a \in A\}$ is bounded by the number M referred to in the Banach-Steinhaus theorem.

Now we can use the resonance theorem to get a nice result on the limit of a sequence of bounded linear transformations.

IV.6 Theorem: Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear transformations from a Banach space X to a normed linear space Y. Suppose that $\lim_{n\to\infty} T_n x = Tx$ exists in Y for each $x \in X$. Then T is also a bounded linear transformation from X into Y, and

$$\|T\| \leq \lim_{n\to\infty} \inf \|T_n\|.$$

Proof: Since $\{T_n x\}_{n=1}^{\infty}$ has a limit at each $x \in X$, $\{\|T_n x\|\}_{n=1}^{\infty}$ is bounded at each $x \in X$. Thus, we can invoke the resonance theorem and deduce that the sequence $\{\|T_n\|\}_{n=1}^{\infty}$ is bounded. Now let x and y be in X, and let α and β be scalars. Then

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} (\alpha\, T_n x + \beta\, T_n y) \\ &= \alpha \lim_{\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y \\ &= \alpha\, Tx + \beta\, Ty. \end{aligned}$$

Thus, T is a linear transformation.

Now fix $x \in X$. Since

$$0 \leq \Big| \|Tx\| - \|T_n x\| \Big| \leq \|Tx - T_n x\|$$

and $\|Tx-T_n x\| \to 0$ as $n \to \infty$, $\Big|\|Tx\| - \|T_n x\|\Big| \to 0$ as $n \to \infty$ and we have $\lim\limits_{n\to\infty} \|T_n x\| = \|Tx\|$. Since $\lim\limits_{n\to\infty} \|T_n x\|$ exists,

$$\lim_{n\to\infty} \inf \|T_n x\| = \lim_{n\to\infty} \|T_n x\|.$$

For each n, $\|T_n x\| \le \|T_n\|\|x\|$, so

$$\liminf_{n\to\infty} \|T_n x\| \le \liminf_{n\to\infty} \|T_n\|\,\|x\| = \|x\| \liminf_{n\to\infty} \|T_n\|.$$

Therefore,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| = \liminf_{n\to\infty} \|T_n x\|$$

$$\le \|x\| \liminf_{n\to\infty} \|T_n\|.$$

Since $\{\|T_n\|\}_{n=1}^{\infty}$ is bounded,

$$\liminf_{n\to\infty} \|T_n\| < \infty,$$

and thus T is bounded, $\|T\| \le \liminf\limits_{n\to\infty} \|T_n\|$. This completes the proof.

Note that in general it would not have been enough to hypothesize that each $\{T_n x\}_{n=1}^{\infty}$ be a Cauchy sequence, since Y, in general, is not complete. If Y is a Banach space, it suffices to require that each $\{T_n x\}_{n=1}^{\infty}$ be a Cauchy sequence. The proof is then left unchanged after invoking the completeness of Y.

Section 3: An Application of the Banach-Steinhaus Theorem

To demonstrate the power of the tools we have been developing, we shall now use them to investigate, and answer, a classical question in the theory of Fourier series. Before going into the matter at hand, however, we shall mention some terminology and general background material.

Let T denote the unit circle in the complex plane, i.e.,

$$T = \{z: |z| = 1\}.$$

If F is a complex-valued function on T and f is a complex-valued function on E^1 given by $f(t) = F(e^{it})$, then f is periodic of period 2π. Similarly, if f is a 2π-periodic complex-valued function of a real variable, there exists an F on T so that $f(t) = F(e^{it})$ for all real t. Thus, the space of all continuous complex-valued 2π-periodic functions of a real variable will be called $C(T)$. For f simply complex-valued and 2π-periodic, we define

$$\|f\|_p = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p \, dt\right]^{1/p}$$

and

$$L^p(T) = \{f: \|f\|_p < \infty\},$$

$1 \leq p < \infty$. Note that the measure being used here is Lebesgue measure divided by 2π. For f and g in $C(T)$, we define

$$d(f,g) = \max \{|f(x)-g(x)|: -\pi \leq x \leq \pi\}.$$

For f and g in $L^p(T)$, $1 \leq p < \infty$, we define

$$d(f,g) = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)-g(t)|^p \, dt\right]^{1/p} .$$

Wherever there might arise a chance of confusing these metrics, we shall be quite explicit. If we define the norm in $C(T)$ by

$$\|f\| = \max\ \{|f(x)|: -\pi \leq x \leq \pi\},$$

then $C(T)$ is a Banach space using pointwise addition. Each $L^p(T)$, with the associated $\|\cdot\|_p$, yields a Banach, again using pointwise addition and identifying functions differing only on sets of Lebesgue measure zero.

For each positive integer n, define $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$. Now for $f \in C(T)$ and fixed x, define

$$s_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt.$$

$s_n(f;x)$ is called the nth partial Fourier sum of f at x. It is known that $s_n(f;x)$, considered as a function of x, approaches f in the $L^2(T)$ norm (see [9], pp. 91-92). Thus, there is a subsequence of $\{s_n(f;x)\}_{n=1}^{\infty}$ which converges pointwise to f, except possibly on a set of Lebesgue measure zero. This gives rise to the following question: Is it true for every $f \in C(T)$ that the Fourier series of f converges to f(x) at every point x?

The answer, it turns out, is no. Fix x, and for $f \in C(T)$ and n a positive integer, define

$$T_n f = s_n(f;x).$$

Now each T_n is clearly a linear transformation from C(T) to the Banach space of complex numbers. Now

$$\begin{aligned}\|T_n f\| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(t)\, D_n(x-t)dt\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\ |D_n(x-t)|dt \\ &\le \frac{1}{2\pi}\left(\max\{|f(t)|: -\pi \le t \le \pi\}\right)\int_{-\pi}^{\pi} |D_n(x-t)|dt \\ &= \|f\|\,\|D_n\|_1,\end{aligned}$$

where the norm of f is that of C(T). Since $\|T_n f\| \le \|f\|\|D_n\|_1$, $\|T_n\| \le \|D_n\|_1$, for every positive integer n. In particular, this says that each T_n is a bounded linear transformation.

Now we show that $\|T_n\| = \|D_n\|_1$ for each n. Note that, although it might appear otherwise, each D_n is a real-valued function, as we shall show later. Fix n and define a function g by g(t) = 1 if $D_n(x-t) \ge 0$ and g(t) = -1 if $D_n(x-t) < 0$. Find a sequence $\{f_j\}_{j=1}^{\infty}$ in C(T) such that for each $j \ge 1$, $-1 \le f_j(t) \le 1$ for all t, and such that $f_j \to g$ pointwise. By Lebesgue's dominated convergence theorem,

$$\lim_{j\to\infty} T_n f_j = \lim_{j\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(t)\, D_n(x-t)dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, D_n(x-t)dt = \|D_n\|_1 .$$

Since for every j, $\|f_j\| \leq 1$, it follows that $\|T_n\| \geq \|D_n\|_1$. Since we already have

$$\|T_n\| \leq \|D_n\|_1,$$

this gives

$$\|D_n\|_1 = \|T_n\| .$$

Now we go back to the definition $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$. This gives

$$e^{\frac{it}{2}} D_n(t) = \sum_{k=-n}^{n} e^{i(k+1/2)t}$$

and

$$e^{-\frac{it}{2}} D_n(t) = \sum_{k=-n}^{n} e^{i(k-1/2)t}.$$

Hence

$$\left[e^{\frac{it}{2}} - e^{-\frac{it}{2}}\right] D_n(t) = e^{i(n+1/2)t} - e^{i(-n-1/2)t}.$$

Thus

$$(2i \sin t/2)D_n(t) = 2i \sin(n+1/2)t,$$

and

$$D_n(t) = \frac{\sin[(n+1/2)t]}{\sin(t/2)} .$$

Since $|\sin t/2| \leq |t/2|$ for all t,

$$\|D_n\|_1 \geq \frac{2}{\pi} \int_0^{\pi} \left|\frac{\sin(n+1/2)t}{t}\right| dt = \frac{2}{\pi} \int_0^{(n+1/2)\pi} \left|\frac{\sin t}{t}\right| dt$$

$$> \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin t| dt$$

$$= \frac{4}{\pi} \sum_{i=1}^{n} \frac{1}{k} .$$

But $\sum_{i=1}^{n} \frac{1}{k} \to \infty$ as $n \to \infty$. Hence $\|D_n\|_1 \to \infty$, and thus $\|T_n\| \to \infty$ as $n \to \infty$. Thus the sequence $\{\|T_n\|\}_{n=1}^{\infty}$ is not bounded, and the second alternative in the Banach-Steinhaus theorem applies. We conclude that for each real x there is a set $E_x \subset C(T)$, which is a dense G_δ set in C(T), such that, if $f \in E_x$, then $\sup \{|s_n(f;x)|: n=1,2,\cdots\} = \infty$. We see it is quite possible to have a continuous 2π-periodic function without an everywhere convergent Fourier series.

Section 4: The Open Mapping Theorem

If $f: X \to Y$ is a function from a topological space X to a topological space Y, we call f an *open mapping* if $f(U)$ is open in Y whenever U is open in X. It should be noted that not every continuous function is an open mapping, as can easily be seen by the following example. Let $X = Y = E^1$ and let $f: X \to Y$ be given by $f(x) = x^2$. Now let $U = (-1,1)$. U is open in X, but $f(U) = [0,1)$ is not open in Y. It does, however, turn out that every bounded linear transformation from one Banach space to another is an open mapping.

IV.7 Theorem (The open mapping theorem): Let X and Y be Banach spaces, and let $T: X \to Y$ be a bounded linear transformation *onto* Y. Then $T(U)$ is open in Y whenever U is open in X.

Proof: Let $U = \{x: x \in X, \|x\| < 1\}$ and $V = \{y: y \in Y, \|y\| < 1\}$. First we show that there exists a number $\delta > 0$ such that $\delta V \subset T(U)$, where $\delta V = \{\delta y: y \in V\}$.

Given $y \in Y$, there exists $x \in X$ such that $Tx = y$, since T is onto. Find an integer k such that $\|x\| < k$. Now $x \in kU$, so $y \in T(kU) = kT(U)$. But $y \in Y$ was arbitrary, so $Y = \bigcup_{k\geq 1} k\,T(U)$. Now Y is a Banach space, so in particular it is complete and we can invoke the Baire theorem to infer that there exists an integer n such that $\overline{n\,T(U)}$ contains a non-empty open set W. Fix n and W. Now every point of W is the limit of a sequence $\{Tx_i\}_{i=1}^{\infty}$ where each x_i is in nU, since $W \subset \overline{n\,T(U)}$.

Now pick $y_o \in W$. Since W is open, we can find $r > 0$ such that $\|y\| < r$, $y \in Y$, implies $y_o + y \in W$. Assume that such an r has been chosen. Now fix $y \in Y$, $\|y\| < r$, and let $\{x_i'\}_{i=1}^{\infty}$ and $\{x_i''\}_{i=1}^{\infty}$ be sequences in nU such that

$$\lim_{i\to\infty} T\, x_i' = y_o \quad \text{and} \quad \lim_{i\to\infty} T\, x_i'' = y_o + y.$$

For each $i \geq 1$, let $x_i = x_i'' - x_i'$. Now for each $i \geq 1$,

$$\|x_i\| = \|x_i''-x_i'\| \leq \|x_i''\| + \|x_i'\| < 2n.$$

Also $\lim_{i\to\infty} T\, x_i = y$.

Now let $y\varepsilon Y$ be arbitrary, $y \neq 0$. Now $y_1 = (r/2\|y\|)y$ has norm $r/2 < r$. Find a sequence $\{x_i'\}_{i=1}^{\infty}$ in X so that $\|x_i'\| < 2n$ for each $i \geq 1$ and $Tx_i' \to y_1$. Letting $x_i = (2\|y\|/r)x_i'$ for each $i \geq 1$, we have $Tx_i \to y$ and $\|x_i\| = (2\|y\|/r)\|x_i'\| \leq (4n/r)\|y\|$, for each $i \geq 1$. Let $\varepsilon > 0$ be arbitrary and let $\delta = r/4n$. Now we can clearly find $x\varepsilon X$ so that $\|x\| \leq \delta^{-1}\|y\|$ and $\|y-Tx\| < \varepsilon$. Thus, if $\|y\| < \delta$ there is $x\varepsilon X$ so that $\|x\| < 1$ and $\|y-Tx\| < \varepsilon$.

Now fix $y \in \delta V$ and $\varepsilon > 0$. Choose x_1 such that $x_1 \in X$, $\|x_1\| < 1$, and $\|y-T\, x_1\| < 2^{-1}\, \delta\varepsilon$. Let n be positive integer and assume $x_1, x_2, \cdots, x_n$ have been chosen so that $\|y_n\| < 2^{-n}\, \delta\varepsilon$, where $y_n = y - T\, x_1 - T\, x_2 - \cdots - T\, x_n$. Now $\|2n/\varepsilon\ y_n\| < \delta$, so $2^n/\varepsilon\ y_n \in \delta V$. There exists $x_{n+1}' \in X$, $\|x_{n+1}'\| < 1$, such that

$$\|2^n/\varepsilon\ y_n - T\, x_{n+1}'\| < 1/2\ \delta.$$

Let $x_{n+1} = 2^{-n}\, \varepsilon\, x_{n+1}'$. Now $\|y_n - T\, x_{n+1}\| < 2^{-(n+1)}\delta\varepsilon$, and $\|x_{n+1}\| < 2^{-(n+1)}\varepsilon$. The induction is complete and we now have a sequence

$\{x_n\}_{n=1}^{\infty}$ in X such that $\|x_n\| < 2^{-n}\varepsilon$ and

$$\|y - T x_1 - T x_2 - \cdots - T x_n\| < 2^{-n}\varepsilon,$$

for each $n \geq 1$.

For each integer n, let $s_n = x_1 + x_2 + \cdots + x_n$. Since $\|s_{n+1}-s_n\| = \|x_{n+1}\| < 2^{-n}\varepsilon$, $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence. But X is a Banach space, and thus there exists $x \in X$ such that $\lim_{n\to\infty} s_n = x$. Now

$$\|s_n\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_n\| < 1 + \varepsilon \sum_{k=2}^{n} 2^{-k}.$$

Consequently, $\|x\| = \lim_{n\to\infty} \|s_n\| \leq 1 + 1/2\ \varepsilon < 1 + \varepsilon$. Since T is continuous, $Ts_n \to Tx$ as $n \to \infty$. But also $\|y-Ts_n\| < 2^{-n}\varepsilon$ for each $n \geq 1$, so $\|y-Ts_n\| \to 0$ and $Ts_n \to y$. Thus $Tx = y$. Thus, for $y \in \delta V$ we have found $x \in X$, with $\|x\| < 1 + \varepsilon$, such that $Tx = y$. Hence, $\delta V \subset T\big((1+\varepsilon)U\big)$. Thus,

$$\frac{\delta}{1+\varepsilon} V \subset T(U), \quad \text{for every } \varepsilon > 0.$$

Hence, $\bigcup \frac{\delta}{1+\varepsilon} V \subset T(U)$, where the union is taken over all $\varepsilon > 0$. But $\bigcup \frac{\delta}{1+\varepsilon} V = \delta V$, and we have found a number $\delta > 0$ such that $\delta V \subset T(U)$.

Now let $A \subset X$ be an open set and let $B = T(A) \subset Y$. Pick $y_o \in B$. There exists $x_o \in A$ such that $T x_o = y_o$. Since A is open we can find a number $\varepsilon > 0$ such that $\|x\| < \varepsilon$ implies $x_o + x \in A$.

Now $\delta V \subset T(U)$, so $\delta\varepsilon V \subset \varepsilon T(U) = T(\varepsilon U)$. Now

$$y_o + \delta\varepsilon V \subset T\,x_o + T(\varepsilon U) \subset T(A) = B,$$

where

$$y_o + \delta\varepsilon V = \{y_o+y\colon\ y \in \delta\varepsilon V\}.$$

Thus, $\|y\| < \delta\varepsilon$ implies that $y_o+y \in B$, and B is open. This completes the proof.

This theorem gives an immediate corollary about the continuity of inverses of bounded linear transformations, when such inverses exist.

IV.7 Theorem: Let X and Y be Banach spaces, and let $T\colon X \to Y$ be a one-to-one bounded linear transformation from X *onto* Y. Then $T^{-1}\colon Y \to X$ is a bounded linear transformation.

Proof: Suppose $Tx = \beta y$. Then $T(1/\beta\ x) = y$ and $T^{-1}y = 1/\beta\ x$, $\beta T^{-1}y = x$. But also $T^{-1}(\beta y) = x$, and hence $\beta T^{-1}y = T^{-1}(\beta y)$. Now suppose that $T\,x_1 = y_1$ and $T\,x_2 = y_2$. Then $T(x_1+x_2) = y_1 + y_2$, and therefore $T^{-1}(y_1+y_2) = x_1 + x_2$. Also $T^{-1}\,y_i = x_i$, $i = 1,2$. Thus,

$$T^{-1}(y_1+y_2) = T^{-1}y_1 + T^{-1}y_2.$$

It follows that T^{-1} is linear.

To prove that T^{-1} is bounded, we show that T^{-1} is continuous. Let $U \subset X$ be open. Now $T(U)$ is open, by the open mapping theorem. Since $(T^{-1})^{-1} = T$, the proof is complete.

Section 5: Graphs of Linear Transformations

A famous theorem of Banach tells us how we can sometimes look at the graph of a transformation and from it deduce information about the transformation itself. To get to this theorem we must make several preliminary remarks as background.

Let X and Y be Banach spaces, and let X x Y be their cartesian product. For (x_1,y_1) and (x_2,y_2) in X x Y we define $(x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2)$ and $\alpha(x_1,y_1) = (\alpha x_1, \alpha y_1)$, where α is any scalar. Now X x Y is a linear space since X and Y are both linear spaces. Now we put a topology on X x Y. Let $U \subset X \times Y$. We say U is open if and only if for each $(x,y) \in U$ there exists an open set $V_x \subset X$ and an open set $V_y \subset Y$ such that $x \in V_x$, $y \in V_y$, and $V_x \times V_y \subset U$. This determines a topology on X x Y. We can also define a norm on X x Y by setting $\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$. It should be noted that the metric induced by this norm generates the same topology on X x Y as we defined above. Now we come to a very useful bit of information.

IV.8 Proposition: If X and Y are Banach spaces, then X x Y is a Banach space.

Proof: That X x Y is a normed linear space is clear from our earlier remarks. We now show X x Y is complete.

Let $\{(x_n,y_n)\}_{n=1}^{\infty}$ be a Cauchy sequence in X x Y. Let $\varepsilon > 0$ be given. There exists a positive integer N such that $n > N$ and $m > N$ imply $\|(x_n,y_n) - (x_m,y_m)\| < \varepsilon$. But

$$\|(x_n,y_n) - (x_m,y_m)\| = (\|x_n-x_m\|^2 + \|(y_n-y_m\|^2)^{1/2}.$$

Thus, $\|x_n - x_m\| \leq \|(x_n,y_n) - (x_m,y_m)\| < \varepsilon$ when $n > N$ and $m > N$, and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Similarly, $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Let x_o and y_o be the respective limits of these sequences. There exists N_1 such that $n > N_1$ implies $\|x_n - x_o\| < \varepsilon/\sqrt{2}$, and there exists N_2 such that $n > N_2$ implies $\|y_n - y_o\| < \varepsilon/\sqrt{2}$. Let $N = \max\,\{N_1,N_2\}$. Now if $n > N$ we have

$$\begin{aligned} \|(x_n,y_n) - (x_o,y_o)\| &= (\|x_n - x_o\|^2 + \|y_n - y_o\|^2)^{1/2} \\ &< (\varepsilon^2/2 + \varepsilon^2/2)^{1/2} \\ &= \varepsilon \;, \end{aligned}$$

and thus

$$\lim_{n\to\infty} (x_n,y_n) = (x_o,y_o).$$

This completes the proof.

Now suppose X and Y are Banach spaces, and $T\colon D(T) \to Y$ is a linear transformation, where $D(T) \subset X$ is a linear subspace of X. We define G(T), *the graph of T,* by

$$G(T) = \{(x,Tx)\colon x \in D(T)\}.$$

Since T is linear and D(T) is a linear subspace of X, G(T) is a linear subspace of X x Y.

IV.9 Definition: Let X, Y, T, and G(T) be as in the above discussion. Then T is called a *closed linear transformation* if and only if G(T) is a closed linear subspace of X x Y.

IV.10 Definition: Let X, Y, T, and G(T) be as in the above discussion. Then T is called a *closable linear transformation* if there exists a linear transformation S such that $G(S) = \overline{G(T)}$.

Since this definition of closability for linear operators does not appear to be very operational, it is nice to have a somewhat more workable criterion.

IV.11 Proposition: If X and Y are Banach spaces and $T: D(T) \to Y$ is a linear transformation, where D(T) is a linear subspace of X, then T is closable if and only if the conditions

$$\text{(i)} \quad \{x_n\}_{n=1}^{\infty} \subset D(T),$$

$$\text{(ii)} \quad \lim_{n\to\infty} x_n = 0, \quad \text{and}$$

$$\text{(iii)} \quad \lim_{n\to\infty} T\, x_n = y,$$

imply

$$\text{(iv)} \quad y = 0.$$

Proof: Suppose T is closable and let S be such that $G(S) = \overline{G(T)}$. Assume conditions (i), (ii), and (iii). Now $(0,y) \in G(S)$, since G(S) is closed. But $(0,0) \in G(S)$, so $y = 0$.

Now suppose conditions (i), (ii), and (iii) imply (iv). We define a transformation S on a domain D(S) as follows:

D(S) = {x: (a) There is at least one sequence $\{x_i\}_{i=1}^{\infty} \subset D(T)$ such that $\lim\limits_{i\to\infty} x_i = x$ and $\lim\limits_{i\to\infty} T\, x_i$ exists.

(b) For every $\{x_i\}_{i=1}^{\infty} \subset D(T)$ such that $\lim\limits_{i\to\infty} x_i = x$, $\lim\limits_{i\to\infty} T\, x_i$ exists.}

Note that condition (a) is to keep condition (b) from being vacuously met. For $x \varepsilon D(S)$, find $\{x_i\}_{i=1}^{\infty} \subset D(T)$ so that $\lim\limits_{i\to\infty} x_i = x$. Define $Sx = \lim\limits_{i\to\infty} T\, x_i$. This limit exists by the definition of D(S). Suppose $\{x_i\}_{i=1}^{\infty} \subset D(T)$ and $\{x_i'\}_{i=1}^{\infty} \subset D(T)$ both have limit $x \varepsilon D(S)$. Now $\lim\limits_{i\to\infty} (x_i - x_i') = 0$. By definition of D(S),

$$\lim_{i\to\infty} (T\, x_i - T\, x_i') = \lim_{i\to\infty} T(x_i - x_i')$$

exists, so by hypothesis

$$\lim_{i\to\infty} T\, x_i = \lim_{i\to\infty} T\, x_i',$$

and Sx is well defined. S and D(S) are clearly linear, so we need only to show that S is closed. This follows since clearly $D(S) \subset \overline{D(T)}$, but $D(T) \subset D(S)$, and thus G(S), if it is closed, will equal $\overline{G(T)}$.

Let $\{(w_n, Sw_n)\}_{n=1}^{\infty}$ be a Cauchy sequence in G(S). Now $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence in D(S). Consider, for the moment, a fixed n. By definition of S and D(S) there exists a sequence $\{x_{in}\}_{i=1}^{\infty}$ in D(T) such that $\lim_{i\to\infty} x_{in} = w_n$ and $\lim_{i\to\infty} T\, x_{in} = Sw_n$. Thus there exists $x_n \in D(T)$ such that $\|x_n - w_n\| < 1/n$ and $\|T\, x_n - Sw_n\| < 1/n$. But n here was arbitrary, so we have such an x_n for every n. Now $\lim_{n\to\infty} x_n = \lim_{n\to\infty} w_n = w$, and $\lim_{n\to\infty} T\, x_n = \lim_{n\to\infty} Sw_n = Sw$, and therefore $(w, Sw) \in G(S)$. Thus, G(S) is closed, and the proof is complete.

Our next proposition is an almost trivial result, but it is of interest.

IV.12 Proposition: Let X and Y be Banach spaces, D(T) a linear subspace of X, and T: D(T) → Y a closed linear transformation. If T^{-1} exists, then T^{-1} is a closed linear transformation.

Proof: Note that

$$G(T^{-1}) = \{(Tx, x) \colon x \in D(T)\}.$$

But $\{(Tx,x) \colon x \in D(T)\}$ is the image of G(T) under the homeomorphism $\phi\colon X \times Y \to Y \times X$ given by $\phi\big((x,y)\big) = (y,x)$. Since G(T) is closed, and homeomorphisms map closed sets onto closed sets, $G(T^{-1})$ is closed. The proof is complete.

We next prove the closed graph theorem of Banach. It should be noted that although Baire's theorem is not directly applied in the proof, the open mapping theorem is used, for which we did need Baire's theorem. Recall that a linear transformation T: X → Y from a Banach space X to a Banach space Y is continuous if and only if it is bounded.

Thus, the open mapping theorem asserts that a continuous linear transformation from a Banach space onto a Banach space is an open mapping.

IV.13 Theorem (Banach's Closed Graph theorem): A closed linear transformation $T: X \to Y$ from a Banach space X to a Banach space Y is continuous.

Proof: The graph $G(T)$ is a closed linear subspace of the Banach space $X \times Y$ and is therefore a Banach space. We define $U: G(T) \to X$ by $U(x,Tx) = x$. U is clearly linear. Now

$$\frac{\|U(x,Tx)\|}{\|(x,Tx)\|} = \frac{\|x\|}{(\|x\|^2 + \|Tx\|^2)^{1/2}} \leq 1$$

for all $(x,Tx) \in G(T)$, $x \neq 0$. Thus, U is bounded, and therefore continuous. U is clearly one-to-one, and consequently U^{-1} exists. Both the open mapping theorem and its corollary apply, and we have that U^{-1} is continuous. The mapping $V: G(T) \to Y$ given by $V(x,Tx) = Tx$ is bounded as was U, and thus V is continuous. Now

$$(VU^{-1})x = V(U^{-1}x) = V(x,Tx) = Tx,$$

and hence $T = VU^{-1}$. Since the composition of two continuous mappings is continuous, this asserts that T is continuous, and the proof is complete.

Although the proof of the closed graph theorem appears to have gone quite easily, it should be realized that a considerable amount of machinery was employed to construct the tools with which the theorem was proved.

Hörmander ([12], p. 79) has derived an interesting result comparing linear transformations with domains in the same Banach space.

IV.14 Theorem (Hörmander): Let X, X_1, and X_2 be Banach spaces. For $i = 1,2$ let T_i be a linear transformation from $D(T_i) \subset X$ to X_i. Assume that T_1 is closed, that T_2 is closable, and that $D(T_1) \subset D(T_2)$. Then there exists a real number C such that

$$\|T_2 x\| \leq C(\|T_1 x\|^2 + \|x\|^2)^{1/2}$$

for all $x \in D(T_1)$.

Proof: The graph $G(T_1) \subset X \times X_1$ is a Banach space, by hypothesis. Now $L\colon G(T_1) \to X_2$ given by $L(x, T_1 x) = T_2 x$ is a linear transformation from the Banach space $G(T_1)$ to the Banach space X_2. We shall show that L is a closed linear transformation.

Suppose that $\{(x_n, T_1 x_n)\}_{n=1}^{\infty}$ converges in $G(T_1)$ and $\{T_2 x_n\}_{n=1}^{\infty}$ converges in X_2. Since T_1 is closed we can find $x \in D(T_1)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} T_1 x_n = T_1 x$. Since $D(T_1)$ $D(T_2)$, $x \in D(T_2)$. Consequently, $T_2 x$ exists. But T_2 is closable, and therefore $\lim_{n\to\infty} T_2 x_n$ exists. Consequently, $\lim_{n\to\infty} T_2 x_n = T_2 x$. Thus, L is closed, and the closed graph theorem tells us L is continuous, and therefore bounded. If $x \in D(T_1)$,

$$\|L(x, T_1 x)\| \leq \|L\| \, \|(x, T_1 x)\|.$$

Now $L(x, T_1 x) = T_2 x$ and

$$\|(x, T_1 x)\| = (\|x\|^2 + \|T_1 x\|)^{1/2}.$$

Thus,

$$\|T_2x\| \leq \|L\|(\|x\|^2 + \|T_1x\|^2)^{1/2}.$$

Putting $C = \|L\|$ makes the proof complete.

CHAPTER V

SOME MEASURE THEORETIC CONSIDERATIONS

Section 1: The Construction of a Complete Metric Space from a Measure Space

Suppose S is a set and $\mathcal{B}$ is a collection of subsets of S. We shall call $\mathcal{B}$ a σ-algebra in S if and only if (i), (ii), and (iii) are satisfied:

(i) $S \in \mathcal{B}$

(ii) If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$

(iii) If $\{B_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{B}$, then $\bigcup_{i\geq 1} B_i \in \mathcal{B}$.

A pair $(S,\mathcal{B})$ consisting of a set S and a σ-algebra $\mathcal{B}$ in S is called a *measurable space*. A measure on (S,B) is defined to be an extended real-valued function m on $\mathcal{B}$ such that if $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ is such that $B_i \cap B_j = \phi$ whenever $i \neq j$, then

$$m\Big(\bigcup_{i\geq 1} B_i\Big) = \sum_{i=1}^{\infty} m(B_i).$$

Note that it is permitted to have ∞ as the value of m here. A triple $(S,\mathcal{B},m)$, consisting of a set S, a σ-algebra $\mathcal{B}$ in S, and a measure m on $\mathcal{B}$, will be called a *measure space*. When and if we make reference to a measure space S, it is understood that there is an underlying σ-algebra $\mathcal{B}$ with a measure m. We shall call a measure space S σ-finite if and only if there is a sequence $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ such that $m(B_i) < \infty$ for every $i \geq 1$, and

$$S = \bigcup_{i \geq 1} B_i.$$

Let $(S,\mathcal{B},m)$ be a σ-finite measure space. Let

$$\mathcal{B}_o = \{B\colon B \in \mathcal{B},\ m(B) < \infty\}.$$

Note that in general $\mathcal{B}_o$ is not a σ-algebra, in fact $\mathcal{B}_o$ is a σ-algebra if and only if $m(S) < \infty$. If $B_1 \in \mathcal{B}_o$ and $B_2 \in \mathcal{B}_o$, we define

$$B_1 \ominus B_2 = (B_1^c \cap B_2) \cup (B_1 \cap B_2^c).$$

$B_1 \ominus B_2$ is called the *symmetric difference* of B_1 and B_2. Since $B_1 \ominus B_2$ is formed only by use of complementation, union, and intersection, it follows that, if $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$, then $B_1 \ominus B_2 \in \mathcal{B}$. If $B_1 \in \mathcal{B}_o$ and $B_2 \in \mathcal{B}_o$, then

$$m(B_1 \ominus B_2) = m(B_1^c \cap B_2) + m(B_1 \cap B_2^c)$$

$$\leq m(B_2) + m(B_1) < \infty,$$

and thus $B_1 \ominus B_2 \in \mathcal{B}_o$. Now for $B_1 \in \mathcal{B}_o$ and $B_2 \in \mathcal{B}_o$, we say B_1 is equivalent to B_2, denoted by $B_1 \sim B_2$, if and only if $m(B_1 \ominus B_2) = 0$. To justify our terminology, we now show that $\sim$ is an equivalence relation.

V.1 Lemma: If we define $B_1 \sim B_2$ to mean $m(B_1 \Theta B_2) = 0$ for B_1 and B_2 in $\mathcal{B}_o$, then $\sim$ is an equivalence relation.

Proof: It is clear that $B \sim B$ for each $B \varepsilon \mathcal{B}_o$, since in $(B \Theta B) = m(\phi) = 0$. If $B_1 \sim B_2$, then $m(B_1 \Theta B_2) = 0$. Hence, $m(B_2 \Theta B_1) = m(B_1 \Theta B_2) = 0$, and $B_2 \sim B_1$. Finally, suppose $B_1 \sim B_2$ and $B_2 \sim B_3$. Assume $B_1 \Theta B_3 \quad (B_1 \Theta B_2) \quad (B_2 \Theta B_3)$. Then $0 \leq m(B_1 \Theta B_3) \leq m(B_1 \Theta B_2) + m(B_2 \Theta B_3) = 0$, and thus $m(B_1 \Theta B_3) = 0$, and $B_1 \sim B_3$. It remains to justify the assumption

$$B_1 \Theta B_3 \quad (B_1 \Theta B_2) \quad (B_2 \Theta B_3).$$

Let $x \varepsilon B_1 \Theta B_3 = (B_1 \theta B_3^c) \bigcup (B_1^c \theta B_3)$. Suppose $x \varepsilon B_1 \cap B_3^c$. If $x \notin B_2$, then $x \varepsilon (B_1 \theta B_2)$. If $x \varepsilon B_2$, then $x \varepsilon (B_2 \theta B_3)$, since $x \notin B_3$. In any case $x \varepsilon (B_1 \theta B_2) \bigcup (B_2 \theta B_3)$. Similarly, if $x \varepsilon B_1^c \bigcap B_3$, we have $x \varepsilon (B_1 \theta B_2) \bigcup (B_2 \theta B_3)$. Thus

$$B_1 \Theta B_3 \quad (B_1 \Theta B_2) \quad (B_2 \Theta B_3),$$

and the proof is complete.

Now let $\mathcal{E}_o$ be the collection of all equivalence classes under the above equivalence relation. For E_1 and E_2 in $\mathcal{E}_o$, define $d(E_1,E_2) = m(B_1 \Theta B_2)$, where B_i is any element of E_i, $i = 1,2$.

V.2 Theorem: The space $(\mathcal{E}_o,d)$, as described above, is a complete metric space.

Proof: We first must show that d is a metric. For any E_1 and E_2 in $\mathcal{E}_o$, it is clear that $d(E_1,E_2) \geq 0$. If $d(E_1,E_2) = 0$, $B_1 \sim B_2$ and

$E_1 = E_2$. By the second part of the equivalence relation proof, $d(E_1,E_2) = d(E_2,E_1)$. Now suppose E_1, E_2, and E_3 are in $\mathcal{E}_o$, and B_1, B_2, and B_3 are respective representative elements. Then,

$$(B_1 \ominus B_3) \subset (B_1 \ominus B_2) \cup (B_2 \ominus B_3),$$

as was shown in the proof of V.1. Hence,

$$\begin{aligned} d(E_1,E_3) &= m(B_1 \ominus B_3) \\ &\leq m(B_1 \ominus B_2) + m(B_2 \ominus B_3) \\ &= d(E_1,E_2) + d(E_2,E_3). \end{aligned}$$

Thus, d is a metric on $\mathcal{E}_o$.

To prove the completeness of the metric space $(\mathcal{E}_o,d)$, use will be made of various basic results on Lebesgue integration. The results used are found, for example, in Rudin [9]. In particular we define

$$L^1(S,\mathcal{B},m) = \{f\colon f \text{ is real-valued on } S, \int_S |f|\, dm < \infty\}.$$

If f_1 and f_2 are in $L^1(S,\mathcal{B},m)$, we define

$$\rho(f_1,f_2) = \int_S |f_1-f_2|dm,$$

and recall that $(L^1(S,\mathcal{B},m),\rho)$ is a complete pseudo-metric space. For $B \in \mathcal{B}$, let χ_B be the characteristic function of B. Now let $\{E_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $(\mathcal{E}_o,d)$, and let B_i be a representative of E_i, $i = 1,2,\cdots$. Then

$$d(E_i,E_j) = m(B_i \ominus B_j) = \int_S |\chi_{B_i} - \chi_{B_j}|dm.$$

Thus $\{\chi_{B_j}\}_{i=1}^{\infty}$ is a Cauchy sequence in $(L^1(S,\mathcal{B},m),\rho)$. This says there is a subsequence $\{\chi_{B_{i_k}}\}_{k=1}^{\infty}$ of $\{\chi_{B_i}\}_{i=1}^{\infty}$ which has a pointwise limit, almost everywhere on S relative to m. Let $f \in L^1(S,\mathcal{B},m)$ be such that

$$\lim_{k\to\infty} \chi_{B_{i_k}}(x) = f(x)$$

when the limit exists and $f(x) = 0$ otherwise. At any point of convergence, f is either 0 or 1 since it is a limit of characteristic functions. Let

$$B = \{x\colon\ x \in S, \quad f(x) = 1\}.$$

Now $f = \chi_B$. Let $\varepsilon > 0$ be given. There exists N_1 such that if $n > N_1$ and $m > N_1$ then $d(E_n,E_m) < \varepsilon/2$. Also there exists N_2 such that $i_k > N_2$ implies

$$\int_S |\chi_{B_{i_k}} - \chi_B|dm < \varepsilon/2.$$

Let $N = \max\{N_1,N_2\}$. Now $d(E_n,E) < \varepsilon$ when $n > N$, where E is the element of E_o determined by B. Thus, $\lim_{n\to\infty} E_n = E$ and (E_o, d) is a complete metric space. This completes the proof.

Section 2: The Vitali-Hahn-Saks Theorem

Now fix a measure space (S,B,m), and suppose λ is a complex-valued function on B such that

$$\lambda(\bigcup_{i\geq 1} B_i) = \sum_{i=1}^{\infty} \lambda(B_i)$$

whenever $\{B_i\}_{i=1}^{\infty}$ is a pairwise disjoint sequence in B. Such a λ is called a *complex measure*. A pairwise disjoint collection $\{B_i\}_{i=1}^{\infty}$ is called *a partition of S* if $S = \bigcup_{i\geq 1} B_i$. The *total variation* of a complex measure λ on S, denoted by $|\lambda|(S)$ is defined by

$$|\lambda|(S) = \sup \{\sum_{i=1}^{\infty} |\lambda(B_i)| : \{B_i\}_{i=1}^{\infty} \text{ is a partition of } S\}.$$

We shall say λ is *absolutely continuous with respect to m*, and write $\lambda << m$, if and only if $\lambda(B) = 0$ whenever $m(B) = 0$. We are now prepared to prove the Vitali-Hahn-Saks theorem.

V.3 Theorem (Vitali-Hahn-Saks): Let (S,B,m) be a measure space and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex measures on B. Suppose that for each n, $|\lambda_n|(S) < \infty$ and $\lambda_n << m$. Assume also that, for each $B \in B$, $\lim_{n\to\infty} \lambda_n(B)$ exists as a complex number, denoted $\lambda(B)$. Then the absolute continuity of the λ_n is *uniform* in the sense that if $\{B_i\}_{i=1}^{\infty}$ is a sequence in B such that $\lim_{i\to\infty} m(B_i) = 0$, then $\lim_{i\to\infty} \lambda_n(B_i) = 0$ uniformly in n. Also, if $m(S) < \infty$, λ is a complex measure on B.

Proof: We first recall that the statement $\lambda << m$ is equivalent to the statement: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $m(B) < \delta$ implies $|\lambda(B)| < \varepsilon$. Now for each n define a function t_n on $\mathcal{E}_o$ by $t_n(E) = \lambda_n(B)$, where B is an arbitrary element of E. Suppose $B_1 \varepsilon E$ and $B_2 \varepsilon E$. Then $m(B_1 \ominus B_2) = 0$, which implies $\lambda_n(B_1) = \lambda_n[B_1 \bigcup (B_1 \ominus B_2)] = \lambda_n[(B_1 \ominus B_2) \bigcup B_2] = \lambda_n(B_2)$. Consequently t_n is a function. Note that, given B_1 and B_2 in $\mathcal{B}_o$,

$$|\lambda_n(B_1) - \lambda_n(B_2)| = |\lambda_n(B_1 \ominus B_2)|,$$

for every n. Thus the absolute continuity of the complex measure λ_n implies the continuity of the function t_n, for every n.

Now let $\varepsilon > 0$ be given. The function $t_k - t_{k+n}$ is continuous for fixed k and n. Thus the set

$$\{E: |t_k(E) - t_{k+n}(E)| \leq \varepsilon/3, \quad E \varepsilon \mathcal{E}_o\}$$

is closed in the metric space $\mathcal{E}_o$. Thus

$$F_k(\varepsilon) = \{E: E \varepsilon \mathcal{E}_o, \ \sup_{n\geq 1} |t_k(E) - t_{k+n}(E)| \leq \varepsilon/3\}$$

$$= \bigcap_{n\geq 1} \{E: E \varepsilon \mathcal{E}_o, \ |t_k(E) - t_{k+n}(E)| \leq \varepsilon/3\}$$

is closed, since it is an intersection of closed sets. For $E \varepsilon \mathcal{E}_o$, with representative B, $\lim_{n\to\infty} \lambda_n(B)$ exists, and hence there exists a positive integer N such that $n > N$ and $m > N$ imply $|\lambda_n(B)-\lambda_m(B)| \leq \varepsilon/3$. Thus, E is in $F_{N+1}(\varepsilon)$. Therefore every $E \in \mathcal{E}_o$ is in some $F_k(\varepsilon)$, i.e., $\mathcal{E}_o = \ \underset{k\geq 1}{\ } F_k(\varepsilon)$. But $\mathcal{E}_o$ is a complete metric space and each $F_k(\varepsilon)$ is closed, and thus there exists k_o such that $F_{k_o}(\varepsilon)$ contains a non-empty open set in $\mathcal{E}_o$. Hence there exists $E_o \in \mathcal{E}_o$ and $\eta > 0$ such that $d(E_o,E) < \eta$ implies $\sup\limits_{n\geq 1} |t_{k_o}(E)-t_{k_o+n}(E)| \leq \varepsilon/3$.

Now suppose $B \in \mathcal{B}_o$ and $m(B) < \eta$. Let $B_1 = B \cup B_o$ and $B_2 = B_o - (B \cap B_o)$, where B_o is an element of E_o. Now

$$\begin{aligned} B_1 - B_2 &= B \cup B_o - [B_o \cap (B^c \cup B_o^c)] \\ &= (B \cup B_o) \cap [B_o^c \cup (B \cap B_o)] \\ &= (B \cup B_o) \cap (B \cup B_o^c) = B. \end{aligned}$$

Let E_i, $i = 1,2$, be the equivalence class generated by B_i, and note that $d(E_i,E_o) < \eta$, $i = 1,2$. Suppose $k > k_o$. Then

$$\begin{aligned} |\lambda_k(B)| &= |\lambda_{k_o}(B) - \lambda_{k_o}(B) + \lambda_k(B)| \\ &\leq |\lambda_{k_o}(B)| + |\lambda_{k_o}(B) - \lambda_k(B)| \\ &= |\lambda_{k_o}(B)| + |\lambda_{k_o}(B_1) - \lambda_k(B_1) + \lambda_k(B_2) - \lambda_{k_o}(B_2)| \\ &\leq |\lambda_{k_o}(B)| + |\lambda_{k_o}(B_1) - \lambda_k(B_1)| + |\lambda_{k_o}(B_2) - \lambda_k(B_2)| \end{aligned}$$

$$\leq |\lambda_{k_o}(B)| + \varepsilon/3 + \varepsilon/3 = |\lambda_{k_o}(B)| + 2\ \varepsilon/3.$$

Let $\tilde{\eta} > 0$ be such that $m(B) < \tilde{\eta}$ implies $|\lambda_{k_o}(B)| < \varepsilon/3$. This can be done by using the absolute continuity of λ_{k_o}. Now let $\delta = \min\,\{\eta,\tilde{\eta}\}$. Now $k \geq k_o$ and $m(B) < \delta$ imply $|\lambda_k(B)| < \varepsilon$. For $i = 1,2,\cdots, k_o - 1$, let $\delta_i > 0$ be such that $m(B) < \delta_i$ implies $|\lambda_i(B)| < \varepsilon$. Let $\delta_o = \min\,\{\delta,\delta_1,\delta_2,\cdots,\delta_{k_o-1}\}$. Now $m(B) < \delta_o$ implies $|\lambda_k(B)| < \varepsilon$ for every $k \geq 1$. Thus the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is uniformly absolutely continuous.

Now let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that $m(B) < \delta$ implies $|\lambda_n(B)| < \varepsilon/2$ for every $n \geq 1$. Now $\lim\limits_{n\to\infty} \lambda_n(B)$ exists, and consequently

$$|\lim_{n\to\infty} \lambda_n(B)| = |\lambda(B)| \leq \varepsilon/2 < \varepsilon.$$

Thus, $m(B) < \delta$ implies $|\lambda(B)| < \varepsilon$.

Next we show that λ is finitely additive, i.e., if $\{B_j\}_{j=1}^{n}$ is a *finite* collection of pairwise disjoint sets, then

$$\lambda(\bigcup_{j=1}^{n} B_j) = \sum_{j=1}^{n} \lambda(B_j).$$

$$\lambda(\bigcup_{j=1}^{n} B_j) = \lim_{k\to\infty} \lambda_k(\bigcup_{j=1}^{n} B_i) = \lim_{k\to\infty} \sum_{j=1}^{n} \lambda_k(B_j)$$

$$= \sum_{j=1}^{n} [\lim_{k\to\infty} \lambda_k(B_j)]$$

$$= \sum_{j=1}^{n} \lambda(B_j).$$

The limit interchange here is valid since the sum is a finite sum. Now suppose $m(S) < \infty$. Let $\{B_j\}_{j=1}^{\infty}$ be a pairwise disjoint sequence in $\mathcal{B}_o$ (= $\mathcal{B}$ since $m(S) < \infty$). Now

$$\lambda(\bigcup_{j\geq 1} B_j) = \lambda\left[(\bigcup_{j=1}^{n} B_j) \cup (\bigcup_{j\geq n+1} B_j)\right]$$

$$= \sum_{j=1}^{n} \lambda(B_j) + \lambda(\bigcup_{j\geq n+1} B_j),$$

for each positive integer n. Thus,

$$\lambda(\bigcup_{j\geq 1} B_j) - \sum_{j=1}^{n} \lambda(B_j) = \lambda(\bigcup_{j\geq n+1} B_j).$$

Now since m is a measure,

$$m(\bigcup_{j\geq 1} B_j) = \sum_{j=1}^{\infty} m(B_j) < \infty ,$$

since $m(S) < \infty$. Since $\sum_{j=1}^{\infty} m(B_j) < \infty$, there exists a positive integer N such that $n > N$ implies $\sum_{j=n}^{\infty} m(B_j) < \delta$. But $\sum_{j=n}^{\infty} m(B_j) = m(\bigcup_{j\geq n} B_j) < \delta$, so $|\lambda(\bigcup_{j\geq n} B_j)| < \varepsilon$. But

$$|\lambda(\bigcup_{j\geq 1} B_j) - \sum_{j=1}^{n-1} \lambda(B_j)| = |\lambda(\bigcup_{j\geq n} B_j)| < \varepsilon.$$

Thus, given $\varepsilon > 0$, we have found N such that $n > N$ implies

$$|\lambda(\bigcup_{j\geq 1} B_j) - \sum_{j=1}^{n-1} \lambda(B_j)| < \varepsilon.$$

Consequently,

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda(B_j) = \sum_{j=1}^{\infty} \lambda(B_j)$$

exists, and

$$\sum_{j=1}^{\infty} \lambda(B_j) = \lambda(\bigcup_{j\geq 1} B_j).$$

This completes the proof.

Now suppose we are given a measurable space $(S,\mathcal{B})$, and a sequence of complex measures $\{\lambda_n\}_{n=1}^{\infty}$ on $\mathcal{B}$. Then it is sometimes possible to construct from $\{\lambda_n\}_{n=1}^{\infty}$ a measure m is such a fashion that the conditions of the Vitali-Hahn-Sahs are imitated.

V.4 Corollary: Let $(S,\mathcal{B})$ be a measurable space, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex measures on $(S,\mathcal{B})$ such that $|\lambda_n|(S) < \infty$ for every $n \geq 1$. If a finite $\lim_{n\to\infty} \lambda_n(B)$ exists for each $B \in \mathcal{B}$, then the countable additivity of λ_n is uniform in n, in the sense that $\lim_{k\to} \lambda_n(B_k) = 0$ uniformly in n for any decreasing sequence $\{B_k\}_{k=1}^{\infty}$ in $\mathcal{B}$ such that $\bigcap_{k\geq 1} B_k = \phi$.

Proof: For each $n \geq 1$, let

$$\mu_n(B) = \frac{|\lambda_n|(B)}{|\lambda_n|(S)}$$

for each $B \in \mathcal{B}$. Now each μ_n is a positive measure bounded by 1 (see [9], p. 118). Define m on $\mathcal{B}$ by

$$m(B) = \sum_{j=1}^{\infty} 2^{-j}\mu_j(B).$$

Now m is countably additive because each μ_j is countably additive. Clearly each λ_j is m-absolutely continuous. Let λ represent the limit of $\{\lambda_n\}_{n=1}^{\infty}$ as before. Now suppose $\{B_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{B}$ such that $B_{k+1} \subset B_k$ for every $k \geq 1$ and $\bigcap_{k\geq 1} B_k = \phi$. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that $m(B) < \delta$ implies $|\lambda(B)| < \varepsilon$. Thus, there exists a positive integer N such that $n > N$ implies $|\lambda_n(B)| < \varepsilon$. For each $n = 1,2,\cdots, N-1$, there exists $\delta_n > 0$ such that $m(B) < \delta_n$ implies $|\lambda_n(B)| < \varepsilon$. Let $\delta_o = \min\{\delta_1,\delta_2,\cdots,\delta_{N-1},\delta\}$. Now $m(B) < \delta_o$ implies $|\lambda_n(B)| < \varepsilon$ for all $n \geq 1$. Thus, we can find K such that $k > K$ implies $m(B_k) < \delta_o$ and now $|\lambda_n(B_k)| < \varepsilon$ for every $n \geq 1$, if $k > K$. This completes the proof.

V.5 Corollary: λ, in the Vitali-Hahn-Sahs theorem is countably additive even if $m(S) = \infty$.

Proof: Construct a measure $\tilde{m}$ as in the proof of V.4, and then invoke V.3.

Earlier work along similar lines, and concerned with Lebesgue measure on [0,1], was done by Vitali and Hahn. The theorem in the generality considered here is mainly due to Saks [10]. This is one of the most important theorems on complex measures, and finds strong and very subtle applications in the study of the weak topology in L^p spaces.

Now suppose $(S,\mathcal{B},m)$ is a measure space, with m σ-finite. Consider the Banach space $L^1(S,\mathcal{B},m)$ determined by the complex-valued functions integrable on S. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^1(S,\mathcal{B},m)$ and $f \in L^1(S,\mathcal{B},m)$, we say $f_n \to f$ *strongly* if $\|f_n - f\|_1 \to 0$, and we say $f_n \to f$ *weakly* if $\lim_{n\to\infty} T(f_n) = T(f)$ for every continuous linear functional T on $L^1(S,\mathcal{B},m)$. A standard result in the theory of L^p spaces asserts that the dual space of $L^1(S,\mathcal{B},m)$ is $L^{\infty}(S,B,m)$, when m is σ-finite. A proof of this result, based on the Radon-Nikodym theorem, is found in Rudin [9], pp. 128-130. (Although it is possible to weaken the hypothesis on m, the result does not hold for every positive measure). In view of the result quoted, weak convergence can be described as follows: $f_n \to f$ weakly if and only if

$$\lim_{n\to\infty} \int_S f_n\, g\, dm = \int_S fg\, dm,$$

for every $g \in L^{\infty}(S,\mathcal{B},m)$.

The Vitali-Hahn-Saks theorem can be used very successfully in the study of weak convergence.

V.6 Theorem: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^1(S,\mathcal{B},m)$, where m is σ-finite. If $\{\|f_n\|_1\}_{n=1}^{\infty}$ is bounded, and if

$$\lim_{n\to\infty} \int_B f_n\, dm$$

exists and is finite for each $B \in \mathcal{B}$, then there exists $f \in L^1(S,\mathcal{B},m)$ so that $f_n \to f$ weakly.

Proof: For each positive integer n, define λ_n on $\mathcal{B}$ by

$$\lambda_n(B) = \int_B f_n \, dm.$$

By the elementary properties of the integral, we see that each λ_n is a complex measure and that $\lambda_n << m$ for each $n \geq 1$. By hypothesis $\lim_{n\to\infty} \lambda_n(B) = \lim_{n\to\infty} \int_B f_n \, dm$ exists for each $B \in \mathcal{B}$. Thus, the Vitali-Hahn-Saks theorem applies, and the set function λ on $\mathcal{B}$ given by

$$\lambda(B) = \lim_{n\to\infty} \lambda_n(B)$$

is a complex measure, $\lambda << m$. Thus, by the Radon-Nikodym theorem, which applies since m is σ-finite, there exists $f \in L^1(S,\mathcal{B},m)$ so that

$$\lambda(B) = \int_B f \, dm$$

for each $B \in \mathcal{B}$. Thus,

$$\lim_{n\to\infty} \int_B f_n \, dm = \int_B f \, dm$$

for each $B \in \mathcal{B}$. If χ_B denotes the characteristic function of B, this says

$$\lim_{n\to\infty} \int_S f_n \chi_B \, dm = \int_S f \chi_B \, dm.$$

By the linearity of both the limit and the integral, it follows immediately that

$$\lim_{n\to\infty} \int_S f_n h \, dm = \int_S f h \, dm$$

for any complex-valued simple function h. Let $g \in L^\infty(S,\mathcal{B},m)$. We may suppose g is bounded on S, by altering g on a set of measure zero if necessary. Recall that given $\eta > 0$ we can find a complex-valued simple function h so that $\|g-h\|_\infty < \eta$.

Let $\varepsilon > 0$ be given and let $g \in L^\infty(S,\mathcal{B},m)$. By hypothesis, $\{\|f_n\|_1\}_{n=1}^\infty$ is bounded. Let $M > 0$ be so that $\|f_n\|_1 < M$ for every $h \geq 1$. Choose a complex-valued simple function h so that

$$\|g-h\|_\infty < \frac{\varepsilon}{3 \max \{M,\|f\|_1\}} .$$

Now find N so that $n > N$ implies

$$\left|\int_S f_n h \, dm - \int_S f h \, dm\right| < \varepsilon/3.$$

This can be done since h is a simple function. Now $n > N$ implies

$$\left|\int_S f_n g \, dm - \int_S fg \, dm\right|$$

$$\le \left|\int_S f_n(g-h)dm\right| + \left|\int_S (f_n-f)h\, dm\right| + \left|\int_S f(h-g)dm\right|$$

$$\le \|g-h\|_\infty \|f_n\|_1 + \left|\int_S (f_n h - fh)dm\right| + \|f\|_1 \|h-g\|_\infty$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus,

$$\lim_{n\to\infty} \int_S f_n g\, dm = \int_S f\, g\, dm.$$

But $g \in L^\infty(S,\mathcal{B},m)$ was arbitrary, so the proof is complete.

The Vitali-Hahn-Saks theorem has a number of applications in analysis, and it usually concerns results of considerable depth and subtlety. One such result, which we state here without proof, is the following: Suppose $f_n \to f$ weakly in $L^1(S,\mathcal{B},m)$, where m is a σ-finite measure. Then $f_n \to f$ strongly if and only if $f_n \to f$ in measure on every set B with $m(B) < \infty$. In this result the necessity assertion is trivial since $f_n \to f$ strongly certainly implies $f_n \to f$ in measure on S. The sufficiency assertion is the deep one, and this is where the Vitali-Hahn-Saks theorem is used.

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